- **LIMIT THEOREMS**:

- **Class # 1**:

- **DEF**: $\mathcal{F} \subseteq 2^{\Omega}$ is a σ-algebra if

  (a): $\Omega \in \mathcal{F}$

  (b): if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

  (c): if $\{A_n\}_{n=1}^{\infty} \in \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

[REMARK: $\{A_n\}_1^{\infty} \in \mathcal{F} \Rightarrow \bigcap_1^{\infty} A_n \in \mathcal{F}$, by De Morgen's Laws].

- $(\Omega, \mathcal{F})$ with $\mathcal{F}$ a σ-algebra is a MEASURABLE SPACE. For such a pair, μ is a <u>countably additive measure</u> if $\mu: \mathcal{F} \to [0, \infty]$ satisfies

  (a): $\mu(A) \geq \mu(\emptyset) = 0$, $\forall A \in \mathcal{F}$

  (b): $\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n)$, $\forall$ countable, <u>DISJOINT collection</u> of sets in $\mathcal{F}$.

- If $\mu(\Omega) = 1$, then we call μ a PROBABILITY MEASURE.
- If (b) holds only for FINITE collections, μ is FINITELY ADDITIVE.
- If $\mu(\Omega) < \infty$, then we say μ is finite.
- μ is σ-finite if $\exists$ countable collection B_n covering Ω s.t. $\mu(B_n) < \infty$, $\forall n$.
- $(\Omega, \mathcal{F}, \mu)$ is called a MEASURABLE SPACE (PROBABILITY SPACE if $\mu(\Omega) = 1$).
- Use $\mathbb{P}$ for probability measure
- EX:
  - $A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$.
  - $A \subseteq \bigcup_n A_n \Rightarrow \mathbb{P}(A) \leq \sum_n \mathbb{P}(A_n)$.
  - If $A_n \uparrow A$ denotes $A_1 \subseteq A_2 \subseteq \ldots \subseteq A = \bigcup_n A_n$ then $\mathbb{P}(A_n) \nearrow \mathbb{P}(A)$ (Continuity from below)
  - If $A_n \downarrow A$ ($A_1 \supseteq A_2 \supseteq \ldots \supseteq A = \bigcap_n A_n$), then $\mathbb{P}(A_n) \searrow \mathbb{P}(A)$ (Cont. from above)

[First 3 properties hold for any measure, last property may fail (it holds if $\mu(A_n) < \infty$ for some n) EG: If $A_n = [n, +\infty)$ with Lebesgue or Counting measure].

- If μ is finitely add. and $\forall B_n \in \mathcal{F}$ $B_n \downarrow \phi$, $\mu(B_n) < \infty \Rightarrow \mu(B_n) \downarrow 0$, then μ is countably additive [only if $\mu(\Omega) < \infty$] ?

Counter Example:

$$\begin{cases} \mu(A) = 0 & \text{if } |A| < \infty \\ \mu(A) = +\infty & \text{otherwise} \end{cases}$$

μ is finitely additive [and cont's from above] but not countably additive. ? [Take $\Omega = \mathbb{N}$, $\mathcal{F} = 2^{\Omega}$, $A_i = \{i\}$ . . .]

- If Ω is countable, then we will take $\mathcal{F} = 2^{\Omega}$. Assign $p_\omega \geq 0$ to every $\omega \in \Omega$ s.t. $\sum_{\omega} p_\omega = 1$. ← DOES NOT WORK FOR Ω UNCOUNTABLE CANNOT CONSTRUCT IT THIS WAY.

• **REMARK:** $\mathbb{P}$ is **non-atomic** if

$\mathbb{P}(A) > 0 \Rightarrow \exists\, B \subset A$ s.t.

$0 < \mathbb{P}(B) < \mathbb{P}(A)$.

Observation: (i) If $\mathbb{P}$ is non-atomic (as $\mathcal{L}[0,1]$ for example), then

$\forall$ ~~$\mathbb{P}$~~ A with $\mathbb{P}(A) > 0$,

$\forall \varepsilon > 0 \quad \exists\, B \subseteq A$ s.t.

$0 < \mathbb{P}(B) < \varepsilon$?

(ii): Moreover, $\forall\, 0 < \delta < \mathbb{P}(A)$

$\exists\, B \subset A : \mathbb{P}(B) = \delta$. ?

• **EXERCISE:** $\bigcap$ of (possibly uncountably many) σ-algebra is still a σ-algebra.

↳ This suffers the following definition:

• **DEF:** $\{A_\alpha : \alpha \in I\}$ subsets of Ω, I possibly uncountable, define

$\sigma(\{A_\alpha : \alpha \in I\})$ = smallest σ-algebra $\mathcal{F}$ s.t. $A_\alpha \in \mathcal{F}\ \forall \alpha \in I$.

Namely, $= \bigcap \{\mathcal{G} : \mathcal{G} \subseteq 2^\Omega$ a σ-alg, $A_\alpha \in \mathcal{G}\ \forall \alpha \in I\}$

- **DEF**: If $\Omega = S$ topological space, then the Borel σ-algebra on S is defined as

$$\mathcal{B}_S = \sigma\left(\{O \subset S \text{ open}\}\right).$$

(on $\mathbb{R}$ just use $\mathcal{B}$).

- Comment: $\mathcal{B}$ is also $\sigma\left(\{(a,b) : a<b \in \mathbb{Q}\}\right)$

and also $\sigma\left(\{(a,b] : a<b \in \mathbb{Q}\}\right)$

" " $\sigma\left(\{[a,b) : a<b \in \mathbb{R}\}\right)$

" " $\sigma\left(\{(-\infty, b] : b \in \mathbb{Q}\}\right)$

How can we verify this? If $\mathcal{A} \subseteq \mathcal{F}$ σ-algebra, then $\sigma(\mathcal{A}) \subseteq \mathcal{F}$. So, in order to show $\sigma(\mathcal{A}) = \sigma(\mathcal{A}')$, it suffices to show $\mathcal{A} \subseteq \sigma(\mathcal{A}')$ and $\mathcal{A}' \subseteq \sigma(\mathcal{A})$.

- **EXAMPLE:** $\mathcal{F} = \sigma\left(\{A_\alpha\}_{\alpha \in I}\right)$, I uncountable

Then, $\forall\, B \in \mathcal{F}$, $\exists$ countable subcollection $\{A_n\}_{n \geq 1}$ s.t.

$$B \in \sigma\left(\{A_n\}_{n \geq 1}\right)$$

~~PROOF~~

PROOF: $\mathcal{M} = \{ B \subseteq \Omega^{\mathcal{F}} \text{ with the desired property} \}$

Want to show $\mathcal{M} = \mathcal{F}$.

(Clearly, $\mathcal{M} \subseteq \mathcal{F}$.)

OBS: $\mathcal{M}$ is a σ-algebra itself.

(countable union of countable sets is countable)

Clearly, $\{A_\alpha\}_{\alpha \in I} \subseteq \mathcal{M}$

$$\Rightarrow \underbrace{\sigma\left(\{A_\alpha\}_{\alpha \in I}\right)}_{= \mathcal{F}} \subseteq \mathcal{M}$$

■

- **Lebesgue Measure** λ is the unique measure s.t. $\lambda(A) = \sum_{k=1}^{n} b_k - a_k$ for

$$A = \bigcup_{k=1}^{n} (a_k, b_k]$$

(where $a_1 < b_1 < a_2 < b_2 < \dots < b_n$ and $n < \infty$)

- Uniform probability distribution on $[0,1]$ is just λ restricted to $\mathcal{B}_{[0,1]}$ (sub-σ-algebra of $\mathcal{B}$).

- **THM.** (Caratheodory's extension theorem)

If $\mu_0 : \mathcal{A} \to [0, \infty]$ is a countably additive measure on an algebra $\mathcal{A}$, then $\exists!$ μ on $(\Omega, \sigma(\mathcal{A}))$ s.t. $\mu|_{\mathcal{A}} = \mu_0$ (uniqueness holds only for $\mu_0(\Omega) < \infty$. Actually, uniqueness extends to σ-finite.)

- Here: countably add. on an algebra means $\mu(A_n) \downarrow 0$, $\forall$ collection $A_n \downarrow \emptyset$ (equiv: you can ask $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$ hold only when $\bigcup_n A_n \in \mathcal{A}$, $\mathcal{A}$ the algebra).

- For us: $\mathcal{A}$ = $\forall$ set $A = \bigcup_{k=1}^{n} (a_k, b_k]$ (for $a_1 < b_1 < a_2 < b_2 < \dots$ and $n < \infty$).
  - $\mu_0(A) = \sum b_k - a_k$ is countably additive (this uses Heine-Borel).

NEXT TIME:

- π-system ($\mathcal{P}$ closed under intersections)
- λ-system ($\mathcal{L}$ closed under monotone limits & has $A \setminus B$ $\forall A, B \in \mathcal{L}$)
- Dynkin's π-λ Thm

- $X : \Omega \to (S, \Sigma)$ is a R.V. if $X^{-1}(B) \in \mathcal{F}$ $\forall$ $B \in \Sigma$.