Class #2:

- $\mathcal{P}$ a collection of sets, $\mathcal{L}$ a collection of sets.
- **π-system:** $I, J \in \mathcal{P} \Rightarrow I \cap J \in \mathcal{P}$
- **λ-system:** $\Omega \in \mathcal{L}$
  - $A, B \in \mathcal{L}$, $A \subseteq B \Rightarrow B \setminus A \in \mathcal{L}$
  - $A_i \uparrow A$, $A_i \in \mathcal{L}$, $\Rightarrow A \in \mathcal{L}$.
- **NOTE:** $\mathcal{A}$ algebra $\Rightarrow \mathcal{A}$ is a π-system.
- **PROPOSITION:** $\mathcal{F}$ is a π-system and a λ-system $\iff \mathcal{F}$ is a σ-alg.
- **THM:** (Dynkin's π-λ THM)
  $\mathcal{P}$ a π-system, $\mathcal{L}$ a λ-system, $\mathcal{P} \subseteq \mathcal{L}$,
  then $\sigma(\mathcal{P}) \subseteq \mathcal{L}$.
  (**PROOF IDEA:** smallest λ-system containing $\mathcal{P}$ is a π-system (!). So by PROPOSITION it's a σ-algebra, ...

... so it contains $\sigma(\mathcal{P})$.
but it is also contained in $\mathcal{L}$).

- **COROLLARY**: If μ_1, μ_2 measures on $(\Omega, \mathcal{F})$ s.t.

$$\begin{cases} \cdot\ \mathcal{F} = \sigma(\mathcal{P}) \\ \cdot\ \mu_1 = \mu_2 \text{ on } \mathcal{P} \\ \cdot\ \mathcal{P} \text{ is a } \pi\text{-syst.} \\ \cdot\ \mu_1(\Omega) = \mu_2(\Omega) < \infty \end{cases}$$

then $\mu_1 = \mu_2$ on $\mathcal{F} = \sigma(\mathcal{P})$.

[**CARATHEODORY**: $\mathcal{P}$ is on A, an algebra, which is a π-system.]

PROOF: $\mathcal{L} = \{A \in \sigma(\mathcal{P}) : \mu_1(A) = \mu_2(A)\}$

By assmpt: $\mathcal{P} \subseteq \mathcal{L}$, $\mathcal{P}$ a π-system.

(i): If $A \subseteq B$ & $A, B \in \mathcal{L}$, then

$$\mu_1(B \setminus A) = \mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A) = \mu_2(B \setminus A).$$

and $B \setminus A \in \mathcal{L}$.

...

... (ii): Similarly, $A_i \uparrow A$, $A_i \in \mathcal{L}$, then by continuity from below, $A \in \mathcal{L}$.

(i), (ii) $\Rightarrow$ $\mathcal{L}$ is a λ-system. Thus, by Dynkin's π-λ Thm: $\sigma(\mathcal{P}) \subseteq \mathcal{L}$. ∎

- **REMARK:** Alternative to Dynkin's π-λ. Thm would be to use the MONOTONE CLASS THM (see Varadan's Book)

- **DEF:** $\mathcal{M}$ is a Monotone Class if it's closed under monotone limits (closed under monotone increasing unions and monotone decreasing intersections).

- **THM:** (Halmo's Monotone Class Thm): If $\mathcal{A}$ algebra, $\mathcal{M}$ monotone class and $\mathcal{A} \subseteq \mathcal{M}$, then $\sigma(\mathcal{A}) \subseteq \mathcal{M}$.

CHAPTER # 1.2: RANDOM VARIABLES.

- **DEF**: X a R.V., $X:(\Omega,\mathcal{F})\to(\mathbb{S},\mathcal{S})$, if $X^{-1}(B)\in\mathcal{F}$, $\forall B\in\mathcal{S}$.
- By default: $(\mathbb{S},\mathcal{S})$ will be taken to be $(\mathbb{R},\mathcal{B})$.
- $m\mathcal{F}$ is the set of $(\mathbb{R},\mathcal{B})$-measurable functions.
- **OBSERVATION**: If $\mathcal{J}=\sigma(\mathcal{A})$ and $X^{-1}(A)\in\mathcal{F}$, $\forall A\in\mathcal{A}\Rightarrow X$ is a R.V.
- **EXAMPLE**: (1): $\forall A\in\mathcal{F}$, the function $\mathbb{1}_A$ is a R.V.

  (2): $\forall A_1,\dots,A_n\in\mathcal{F}$, $c_1,\dots,c_n\in\mathbb{R}$, $\sum_{i=1}^{n} c_i\,\mathbb{1}_{A_i}$ is a R.V.

  Called these **SIMPLE FUNCTIONS** (Denoted (SF)).

• **FACT**: (Prop. 1.2.6 in [D])

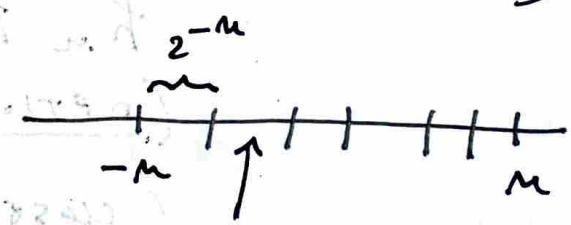
$\forall$ R.V. X $\exists$ a seq. X_n of SF's s.t. $X_n(\omega) \xrightarrow[n\to\infty]{} X(\omega)$, $\forall \omega \in \Omega$.

PROOF: • If $X \geq 0$,

$$X_n(\omega) = n \mathbb{1}_{X(\omega) > n} + \sum_{k=0}^{n2^n - 1} \frac{k}{2^n} \mathbb{1}_{\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]}(X(\omega))$$

(**NOTE**: $X \geq 0$ then $X_n \leq X_{n+1} \leq X$).

⇝ **RANGE of $X(\omega)$:**



If X here, take left-hand point value for X_n

⇝ **Uniform convergence for bounded X.**

• **For general X:** write $X = \underset{\geq 0}{X^+} - \underset{\geq 0}{X^-}$.

• **THM** : (Monotone class Thm).

$\mathcal{H}$ a class of functions $\Omega \longrightarrow \mathbb{R}$.

If (1): $1 \in \mathcal{H}$

(2): $h_1, h_2 \in \mathcal{H}$, $c_1, c_2 \in \mathbb{R}$ $\Rightarrow c_1 h_1 + c_2 h_2 \in \mathcal{H}$ (VECTOR SPACE)

(3): $h_n \in \mathcal{H}$, $h_n \geq 0$ and $h_n \uparrow h$, then $h \in \mathcal{H}$

(OPTIONAL: for h bounded only)

(CLOSED UNDER MONOTONE LIMIT)

Then, if $\mathcal{P}$ π-system s.t. $\mathbb{1}_A \in \mathcal{H}$, $\forall A \in \mathcal{P}$, $\mathcal{H}$ contains ALL MEASURABLE FUNCTIONS w.r.t. $\sigma(\mathcal{P})$

(OPTIONAL: BDD.)

(PROOF IDEA: $\mathcal{L} = \{A \subseteq \Omega : \mathbb{1}_A \in \mathcal{H}\}$ is a λ-system; use Dynkin's π-λ Thm to see that $\sigma(\mathcal{P}) \subseteq \mathcal{L}$).

(⟹ $\mathbb{1}_A \in \mathcal{H}$, $\forall A \in \sigma(\mathcal{P})$. But $\mathcal{H}$ is closed under monotone limits ⟹ ALL $\sigma(\mathcal{P})$-meas. RAND. VARS are in $\mathcal{H}$ ■)

- **DEF**: $X \overset{a.s.}{=} Y$ if $\mathbb{P}\left(\{\omega : X(\omega) \neq Y(\omega)\}\right) = 0$

  (X, Y defined on same probability space).

- **DEF**: $\sigma(X) = \{X^{-1}(B) : B \in \mathcal{J}\}$,

  (also denoted as $\mathcal{F}^X$), for

  $X : (\Omega, \mathcal{F}) \longrightarrow (S, \mathcal{J})$ R.V.

  (σ-alg. generated by X, i.e., smallest σ-algebra w.r.t. which X is measurable).

- **EXERCISE**: (see Thm 1.2.22).

  If X_n are $\overline{\mathbb{R}}$-valued R.V.'s on the same prob. space, then

  $\inf_n X_n$, $\sup_n X_n$, $\underline{\lim}\, X_n$, $\overline{\lim}\, X_n$ are $\overline{\mathbb{R}}$-valued R.V.'s.

- Thus, we include these in the $\overline{\mathbb{R}}$-valued r.v.'s notation in $\mathcal{F}$

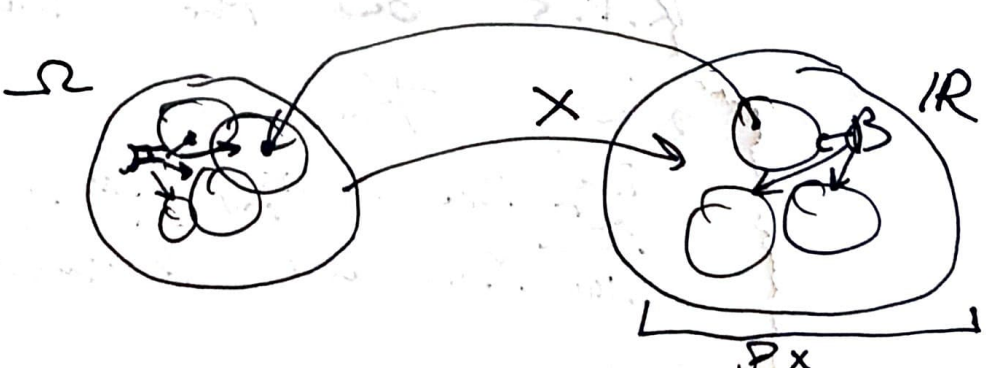
- $\boxed{\text{DEF}}$: $X_n \xrightarrow{A.S.} X_\infty$ if

$$1 = \mathbb{P}\left(\left\{\omega : \underline{\lim}\, X_n(\omega) = \overline{\lim}\, X_n(\omega)\right\}\right),$$

and $X_\infty(\omega) = \overline{\lim}\, X_n(\omega)$
$\left(\overset{\text{or}}{=} \underline{\lim}\, X_n(\omega).\right)$

- $\boxed{\text{DEF}}$: The law $\mathcal{P}_X$ of the r.v. X (on a probability space) is the probability measure on $(\mathbb{R}, \mathcal{B})$ given by

$$\mathcal{P}_X(B) = \mathbb{P}\left(X^{-1}(B)\right), \ \forall \text{ Borel } B$$



(... still need to verify $\mathcal{P}_X$ is a well-def. probability measure on $(\mathbb{R}, \mathcal{B})$).

- **DEF**: $X \overset{d}{=} Y$ IFF $\mathcal{P}_X = \mathcal{P}_Y$.

  ($X = Y$ in distribution or in law).

- **DEF**: The distrib. function F_X of X is
  $F_X(a) = \mathbb{P}(X \le a) = \mathcal{P}_X((-\infty, a])$.

- **THM**: (Thm 1.2.37 [D])

  $F : \mathbb{R} \to [0,1]$ is a distrib. funct. of some R.V. I.F.F.

  (a): F NON-DECREASING

  (b): F RIGHT CONTINUOUS / $F(y) \downarrow F(x)$ as $y \downarrow x$ /

  (c): $F(x) \uparrow 1$ as $x \uparrow \infty$
  $F(x) \downarrow 0$ as $x \downarrow -\infty$.

PROOF: $\cdot$ F distr. fun. of $X \Rightarrow$ (a), (b), (c) trivial

...

- (a), (b), (c) $\Rightarrow$ F is distrib. fn. of R.V. is the nontrivial side:

Skorokhod's Representation:

Define X on $\left((0,1], \mathcal{B}_{(0,1]}, \mu \right)$ Lebesgue meas.

by $X(\omega) = \sup \{ y : F(y) < \omega \}$

$\rightsquigarrow$ $\boxed{\text{Check: } \{\omega : X(\omega) \le x\} = \{\omega : \omega \le F(x)\}}$

$\rightsquigarrow$ and so X is a random var. with the desired distr. fn.

- $\boxed{\text{CHAPTER 1.3 [D]}}$:

- For $(\mathcal{S}, \mathcal{F}, \mu)$, define $\mu(f)$:

(i): $\mu_0(\mathbb{1}_A) = \mu(A)$, $\forall A \in \mathcal{F}$.
(If $\mu = \mathbb{P}$, this means $\mathbb{E}(\mathbb{1}_A) = \mathbb{P}(A)$).

(ii): $\underline{SF^+}$: $\varphi = \sum_1^n c_i \mathbb{1}_{A_i}$, $n < \infty$,

$$\mu_0(\varphi) = \sum_1^n c_i \mu_0(\mathbb{1}_{A_i})$$

(iii): For $f \in m\mathcal{F}^+$:

$$\mu(f) = \sup\{\mu_0(\varphi) : \varphi \in SF^+, \varphi \le f\}$$

(iv): for $f \in m\mathcal{F}$:

write $f = f^+ - f^-$, def

$$\mu(f) = \mu(f^+) - \mu(f^-).$$

if at least one of these is $< \infty$.

- For X on a prob. space:

(i): $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)$

(ii): $\downarrow_{S.F.^+}$

(iii): $\mathbb{E}(X) = \inf\{\mathbb{E}(Y) : Y \in SF^+, Y \ge X\}$

(iv): $\mathbb{E}(X) = \mathbb{E}(X^+) - \mathbb{E}(X^-)$

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