- **Class # 3**

- STANDARD MACHINE (Page 33 of [D]):

(1): Establish property for $h = \mathbb{1}_A$ (every indicator function)

(2): Apply linearity to extend to SF_+

(3): Apply **MON** to extend to mF_+

(4): Write $h = h^+ - h^-$ to extend to mF.

- **MON:** **THM**: (Monotone Conv. Thm).

If $0 \le h_n(s) \nearrow h(s)$, $\forall s \in S$ (pointwise) and $h_n \in mF_+$. Then, $\mu(h_n) \nearrow \mu(h) \le \infty$.

- Using Standard Machine:

Markov's Ineq, Jensen's Ineq., Hölder's Ineq.

- **DEF**: $X_n \xrightarrow{P} X_\infty$ if $\mathbb{P}\{\omega : |X_n(\omega) - X_\infty(\omega)| \geq \varepsilon\} \xrightarrow{n\to\infty} 0$, $\forall\, \varepsilon > 0$.

[equivalent to $|X_n - X_\infty| \xrightarrow{P} 0$]

- If $X_n \xrightarrow{A.E.} X_\infty$, then $X_n \xrightarrow{P} X_\infty$, BUT NOT VICE VERSA.

(i): [($\Longrightarrow$): $A_n = \{\omega : |X_n - X| > \varepsilon\}$

let $S = \{\omega : X_n \not\to X_\infty\}$, $\mathbb{P}(S) = 0$

A_n i.o. $\Rightarrow X_n \not\to X_\infty$

$\Rightarrow \{A_n \text{ i.o.}\} \subseteq S$

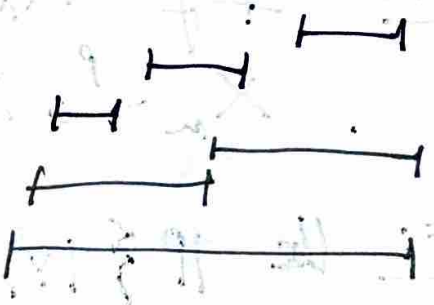
So $0 = \mathbb{P}(S) \geq \mathbb{P}\{A_n \text{ i.o.}\}$

$= \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{m\geq n} A_m\right) \geq \lim_{n\to\infty} \mathbb{P}(A_n)$ ■]

(ii): [(⇐): FALSE, $X_n \xrightarrow{P} X_\infty$

$\not\Rightarrow$ $X_n \xrightarrow{A.E.} X_\infty$

E.g.: Use $((0,1], \mathcal{B}_{(0,1]}, \mu_{(0,1]})$,



f_n: Indicators of $[0, \frac{1}{2}]$,
$[\frac{1}{2}, 1]$, $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$,
$[\frac{2}{3}, 1]$, $[0, \frac{1}{4}]$, ...

so clearly,

$\cdot \mathbb{P}(f_n(\omega) \neq 0) \xrightarrow{n \to \infty} 0$

but $X_n(\omega)$ has NO LIMIT

for any $\omega \in (0, 1]$.

DEF: • X is integrable if $\mathbb{E}|X| < \infty$ (both $\mathbb{E}X^+, \mathbb{E}X^- < \infty$)

• Fix $1 \leq p < \infty$, say that $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$ if $\|X\|_p < \infty$, where $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$.

• **DEF**: $X_n \xrightarrow{L^p} X_\infty$ if $X_n, X_\infty \in L^p$

if $\|X_n - X_\infty\|_p \xrightarrow{n \uparrow \infty} 0$

• **REMARK**: If $X_n \xrightarrow{L^p} X_\infty$, then

$X_n \xrightarrow{P} X_\infty$.

PROOF: Use $\mathbb{P}\{|Y| \geq \varepsilon\} \leq \dfrac{\mathbb{E}|Y|^p}{\varepsilon^p}$

(Markov's) $\uparrow$ $X_n - X_\infty$ ∎

• **EXCORCISE**: (follows from Minkowski's Ineq.)

If $X_n \xrightarrow{L^p} X_\infty$, then

$\mathbb{E}|X_n|^p \longrightarrow \mathbb{E}|X_\infty|^p$, and furthermore, for INTEGER p, also

$\mathbb{E}X_n^p \longrightarrow \mathbb{E}X_\infty^p$

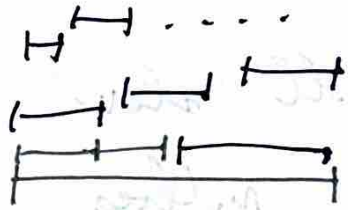
- E.g.: $X_n \xrightarrow{L^p} X_\infty$ $\qquad$ $X_n \xrightarrow{A.S.} X_\infty$

- $\not\Leftarrow$: $U[0,1]$, $X_n(\omega) = n^p \mathbb{1}_{(0,\frac{1}{n}]}$,

$X_n \xrightarrow{A.S.} 0$, $\mathbb{E}\, X_n^p = 1$, so

$X_n \not\xrightarrow{L^p} 0$.

- $\Rightarrow$:



Same

Oscillating function shows

$X_n \not\xrightarrow{A.S.} 0$ but $X_n \xrightarrow{L^p} 0$.

- (READ VITALI'S CONV. THM & UNIFORM INTEGRABILITY; we will revisit it with martingales)

- EXCERCISE: If $X_n \xrightarrow{A.S.} X_\infty$ and $X_n \xrightarrow{L^p} X'_\infty$, then $X_\infty \overset{A.S.}{=} X'_\infty$

- What can we say about $\mathbb{E}X_\infty$ when $X_n \xrightarrow{P} X_\infty$ for instance?

- **THM**: Bounded Convergence Thm (BDD)

If X_n UNIFORMLY BDD, $X_n \xrightarrow{P} X_\infty$,
then $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X_\infty$.

PROOF: We'll show $X_n \xrightarrow{L^1} X$ (i.e., $\mathbb{E}|X_n - X| \to 0$)
This implies $|\mathbb{E}X_n - X| \to 0$.
(By Jensen's / Triangle Ineq.)

W.l.o.g. $|X_n| \le M$ a.s., $X_\infty = 0$, then

$$\mathbb{E}|X_n| = \mathbb{E}|X_n| \mathbb{1}_{\{|X_n| \le \varepsilon\}} + \mathbb{E}|X_n| \mathbb{1}_{\{|X_n| > \varepsilon\}}$$

$$\le \varepsilon + M \underbrace{\mathbb{P}(|X_n| > \varepsilon)}_{\to 0 \text{ since } X_n \xrightarrow{P} 0}.$$

- **FATOU'S LEMMA**: $X_n \geq 0$, $X_n \xrightarrow{P} X$,

Then $\mathbb{E}X \leq \liminf_{n\to\infty} \mathbb{E}X_n$

PROOF: Recall $\mathbb{E}X := \inf\{\mathbb{E}Y : Y \leq X, Y \in SF_+\}$

for $X \in mF_+$. Take $0 \leq Y \leq X$; (BDD)

$X_n \wedge Y = \min\{X_n, Y\}$ is

UNIF. BDD. (by $|Y| < \infty$).

and $X_n \wedge Y \xrightarrow{P} X \wedge Y = Y$

So by BDD:

$$\mathbb{E}Y = \lim_{n\to\infty} \mathbb{E}[X_n \wedge Y]$$

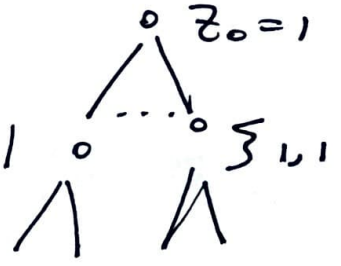
$$\leq \liminf_{n\to\infty} \mathbb{E}X_n.$$

- **EXAMPLE:** (Critical Branching Process).

$Z_0 = 1$; space $(\xi_1, \xi_2, \dots)$ independent non-negative R.V.'s

e.g. $\xi_i \sim \begin{cases} 0, & p = \frac{1}{2} \\ 2, & 1-p = \frac{1}{2} \end{cases}$,

$$Z_n = \sum_{1}^{Z_{n-1}} \xi_{n,i}$$



We'll see that $\mathbb{E}(Z_n) = 1$, $\forall n$

(Generally: $\mathbb{E}\xi < 1$, then $Z_n \xrightarrow{A.S.} 0$; if $\mathbb{E}\xi > 1$, then $Z_n \xrightarrow{A.S.} \infty$)

Fatou's: $\mathbb{E} \lim_{n\to\infty} Z_n \le \underline{\lim_{n\to\infty}} \mathbb{E} Z_n = 1$

EXISTS?

Actually, $\mathbb{E} Z_\infty = 0$

(This will be another example where $X_n \xrightarrow{A.S.} 0$ but $X_n \not\xrightarrow{L^1} 0$).

• **THM**: (MON) If $X_n \geq 0$, $X_n \uparrow X$, then $\mathbb{E}X_n \uparrow \mathbb{E}X$

• **PROOF**: $\forall n$, $\mathbb{E}X_n \leq \mathbb{E}X_{n+1} \leq \mathbb{E}X$, by def. of $\mathbb{E}(\cdot)$. Other side is the NON-TRIVIAL one: USE FATOU'S LEMMA. ∎

• **THM**: (DOM. CONV. THM.)

If $X_n \xrightarrow{P} X_\infty$, $|X_n(\omega)| \leq Y(\omega)$, $\forall \omega \in \Omega$, $\forall n$, for $Y \in L^1$, Then $\mathbb{E}X_n \longrightarrow \mathbb{E}X_\infty$.

PROOF: $Y + X_n \geq 0$, $Y + X_n \xrightarrow{P} Y + X_\infty$

$$\Rightarrow \mathbb{E}[Y + X_\infty] \leq \underline{\lim_{n\to\infty}} \mathbb{E}[Y + X_n]$$

$$\Rightarrow \mathbb{E}X_\infty \leq \underline{\lim} \mathbb{E}X_n$$

(Subtract $\mathbb{E}Y < \infty$) ∎

... <u>OTHER SIDE:</u> Same for $Y - X_n$:

$$\mathbb{E}X_\infty \geq \overline{\lim}\, \mathbb{E}X_n$$

- These theorems were proved in Varhadan's Wory. In [D], MON is proved 1st.