- **Class # 4, LIMIT THMS:**

- **TRANSFORMATION:**

Recall $X: (\Omega_1, \mathcal{B}_1) \longrightarrow (\Omega_2, \mathcal{B}_2)$ is a R.V. if $X^{-1}(B) \in \mathcal{B}_1 \ \forall B \in \mathcal{B}_2$.

Write $\mathbb{P}_X$ denote the <u>law of X</u>: (push forward of $\mathbb{P}$), $\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B)), \forall B \in \mathcal{B}_2$.

- **THM**: (Change of Vars) (THM 1.3.61 in [D]).

For such X, if $f: \Omega_2 \to \mathbb{R}$ measurable on $(\Omega_2, \mathcal{B}_1)$, then $g(\omega_1) := f(X(\omega_1))$ is measurable on $(\Omega_1, \mathcal{B}_1)$ and

- $g \in L^1 \iff f \in L^1$
- $\int_{\Omega_2} f \, d\mathbb{P}_X = \int_{\Omega_1} g \, d\mathbb{P}$.

$\rightsquigarrow_{\infty}$

... [PROOF: STD MACHINE; for $\mathbb{1}_B$,

$$\int_{\Omega_2} f \, d\mathbb{P}_X = \mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B))$$

so can extend to any f ...]

- **PRODUCT SPACES**: Given $(\Omega_1, \mathcal{F}_1), (\Omega_2, \mathcal{F}_2)$

Then product is $\Omega = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) : \omega_i \in \Omega_i\}$

and $\mathcal{F} = \sigma\left(\{ \underbrace{A_1 \times A_2}_{\uparrow} : A_i \in \mathcal{F}_i \} \right)$

MEASURABLE RECTANGLES.

- **EXAMPLE:** $G := \left\{ \bigsqcup_{J=1}^{M} A_1^{(J)} \times A_2^{(J)} : A_i^{(J)} \in \mathcal{F}_i, M < \infty \right\}$ (DISJOINT UNION)

SHOW that G is an ALGEBRA but not a σ-algebra.

$\mathcal{F} := \sigma(G)$.

$\Rightarrow$ ...

• CARATHÉODORY EXTENSION: $\exists!$ $\mathbb{P}$ on $(\Omega, \mathcal{F})$ s.t. $\mathbb{P}\left(\biguplus_{j=1}^{n} A_1^{(j)} \times A_2^{(j)} \right) =$

$$= \sum_{j=1}^{n} \mathbb{P}_1(A_1^{(j)}) \mathbb{P}_2(A_2^{(j)}), \quad \forall \; A_i^{(j)} \in \mathcal{F}_i, \; n < \infty.$$

(Just need to check $\mathbb{P}$ is COUNTABLY ADDITIVE on G) ($\mathbb{P}_i$ measure on $(\Omega_i, \mathcal{F}_i)$).

• E.G.: For 10 independent coin tosses (independence to be def.): This product measure will give us a formulation of the distribution associated with n iid instances of a R.V. $X \rightsquigarrow$ Take the PRODUCT MEASURE of 10 copies of $(\Omega_1, \mathcal{F}_1)$. Works for any $n < \infty$!

→ For $n = \infty$ iid R.V.'s need KOLMOGOROV EXTENSION THM to extend a measure to the space.

- **THM**: (Fubini, Thm 1.4.28 [D]).

$\mu = \mu_1 \times \mu_2$ product measure of σ-finite measures μ_1 on $(X, \mathcal{X})$, $(Y, \mathcal{Y})$.

If $h \in m\mathcal{F}$ ($\mathcal{F}$ = σ-alg. corr. to μ), and $h \geq 0$ OR $\int |h| \, d\mu < \infty$.

then
$$\int_{X \times Y} h \, d\mu = \int_X \left[\int_Y h(x,y) \, d\mu_2(y) \right] d\mu_1(x)$$
$$= \int_Y \left[\int_X h(x,y) \, d\mu_1(x) \right] d\mu_2(y)$$

PROOF: [Std Machine ...].

CHARACTERISTIC FUNCTIONS (§ 3.3 in [D])

X a R.V., its characteristic function $\Phi_X : \mathbb{R} \to \mathbb{C}$ is $\Phi_X(t) = \mathbb{E}\, e^{itX}$.

- **REMINDERS:**

$\Phi(0) = 1$, $|\Phi| \le 1$, Φ UNIFORMLY CONT. on $\mathbb{R}$, Φ POSITIVE SEMIDEFINITE.

$$\sum_{1 \le i,j \le m} \Phi(t_i - t_j)\, \xi_i \bar{\xi}_j \ge 0, \quad \forall \xi \in \mathbb{C}^m,\ t \in \mathbb{R}^m$$

that is, $(\Phi(t_i - t_j))_{1 \le i,j \le m}$ is a POS. SEMIDEF. MATRIX in $\mathbb{C}$.

$$|\Phi(t+\varepsilon) - \Phi(t)| = |\mathbb{E}\, e^{itX}(e^{i\varepsilon X} - 1)|.$$

$$\le \mathbb{E}(|e^{itX}|\,|e^{i\varepsilon X} - 1|)$$

$$= \mathbb{E}\,|e^{i\varepsilon X} - 1| \xrightarrow{\varepsilon \downarrow 0} 0$$

by BDD UNIFORMLY in t, since (t is gone on RHS) $|e^{i\varepsilon X} - 1| \le 2$.

- $\mathbb{E}|X| < \infty$, then $\Phi_X \in C^1$, and $\Phi'(0) = i\,\mathbb{E}X$. More generally, if $\mathbb{E}|X|^n < \infty$, ~~then~~ $\Rightarrow$ $\Phi_X \in C^n$, $\Phi_X^{(n)}(0) = i^n\,\mathbb{E}|X|^n$.

(~~$\Leftarrow$~~ We can have $\Phi \in C^1$ but $\mathbb{E}|X| = \infty$. However, for EVEN n, $\Leftarrow$ HOLDS!

e.g. $\Phi_X''(0) < \infty$, $\Rightarrow$ $\mathbb{E}X^2 < \infty$.)

- PROOF of POSITIVE SEMI-DEF:

$$\sum_{i,j} \Phi_X(t_i - t_j)\,\xi_i \bar{\xi}_j = \sum_{i,j} \xi_i \bar{\xi}_j\, \mathbb{E}\left(e^{i(t_i - t_j)X}\right)$$

$$= \mathbb{E}\left|\sum_{j=1}^{n} \xi_j e^{i t_j X}\right|^2$$

$$\geq 0.$$

- $F_X(x) := \underbrace{P_X(-\infty, x]}$

↑ Law of X, push forward $\in (\mathbb{R}, \mathcal{B})$

- If X has a density, $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

... and

$$\Phi_X(t) = \int_{\mathbb{R}} e^{itx} f_X(x)\,dx$$

(Fourier Transform of f_X - density)

- **Fourier Inversion Suggests:** Recovery f_X from Φ_X.

- **Thm:** (Levy's Inversion Thm) (Thm 3.3.13 in [D])

If X has charact. funct. Φ_X and distrib. function F_X, then

$\forall\ a, b \in$ Cont. Pt (F_X):

$$F(b) - F(a) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \Phi_X(t)\,dt$$

- **Notes:** • For $\forall\ a, b$ this holds if we replace

$$F(a) \leftarrow \frac{F(a) + F(a^-)}{2}, \quad F(b) \leftarrow \frac{F(b) + F(b^-)}{2}$$

- Fourier Inv. Thm. : $f, \hat{f} \in L^1$,

$$\hat{f}(t) = \int_{\mathbb{R}} e^{-2\pi i t x} f(x)\, dx$$

$$f(x) = \int_{\mathbb{R}} e^{2\pi i t x} \hat{f}(t)\, dt$$

- **COROLLARY**: If $\hat{F}_X(t) = \hat{F}_Y(t)$, $\forall t$, then $X \overset{d}{=} Y$.

PROOF: we say $X \overset{d}{=} Y$ if $F_X = F_Y$ on $\underline{C = \text{CONT. PTS.}}$ of F_X & F_Y, hence the above Thm. implies this.

(Also, C is dense in $\mathbb{R}$ and $F_X(x) = \inf\{F_X(z) : z > x, z \in C\}$ right continuous)

- **Proof of Levy's, IDEA:**

Start with RHS, put def. of Φ_X, then Fubini concludes. Checking Fubini's condition should be done with care. (Involves Dirichlet Integral $\left(\int_0^y \frac{\sin(x)}{x}\,dx\right)$).

- **DEF:** X, Y both on $(\Omega, \mathcal{F}, \mathbb{P})$. X, Y independent $\Leftrightarrow$ $P_{(X,Y)}$ on $\mathbb{R}^2$ is the product measure: $P_X \times P_Y$

equiv: $X_1, \dots, X_n$ indep.

$\Leftrightarrow$ $\mathbb{P}\left(\bigcap_1^n \{X_i \in A_i\}\right) = \prod_1^n \mathbb{P}(X_i \in A_i)$,

$\forall A_1, \dots, A_n \in \mathcal{B}$. $\longrightarrow$ and

∞ and $(X_n)_{n=1}^{\infty}$ indep. if $\forall$ finite subcollection is independent.