Class # 5 ; PROBABILITY I

- **CAUCHY:** $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$,

  You showed in HW : $\Phi_X(t) = e^{-|t|}$

  Here: $\Phi'(0)$ doesn't exist

  $\Rightarrow \mathbb{E}|X| = \infty$

- **Observe:** If X, Y are independent, then $Z = X+Y$ satisfies

$$\Phi_Z(t) = \Phi_X(t)\,\Phi_Y(t)$$

(The law of $Z = X+Y$, in general, is $\mu_X * \mu_Y$, i.e.,

$(\mathbb{P} * \mathbb{Q})(A) = \int \mathbb{P}(A-x)\,d\mathbb{Q} = \int \mathbb{Q}(A-x)\,d\mathbb{P}$

by Fubini; so when X, Y are indep:

$$\int e^{itz}\, d(\mathbb{P} * \mathbb{Q}) = \int e^{itx}\, d\mathbb{P} \int e^{ity}\, d\mathbb{Q}$$

$$\| \qquad\qquad \|$$

$$\iint e^{it(x+y)}\, d\mathbb{P}\, d\mathbb{Q} \quad) .$$

* If $X_1, \dots, X_n$ iid, then

$$\Phi_{S_n}(t) = \Phi_{X_1}(t)^n \text{ , where } S_n = \sum_1^n X_i .$$

* $X \sim$ Cauchy, $Y = X$, $Z = X + Y = 2X$,

$\phi_Z(t) = e^{-2t} = \phi_X(t)\phi_Y(t)$, but

X, Y NOT INDEPENDENT.

* EXAMPLE: $\mathbb{P}(X = \pm n) = c \dfrac{1}{n^2 \log(n)}$, $n \geq 2$.

$$\Phi_X(t) = 2c \sum_2^\infty \frac{\cos(nt)}{n^2 \log(n)} \text{ , differentiable at } 0,$$

$$\# X = \sum \frac{1}{n \log(n)} = \infty$$

(Counterexample for $c \dfrac{1}{n + \log^2(n)}$)

- Remember in Lemma 3.3.3 in [D]
  It's an $\Longleftrightarrow$ for EVEN moments.

* If Φ_{S_n} from above was equal to Φ_W, for some R.V. W, we can infer from Levy's Inversion Thm that $W \overset{d}{=} S_n$.
  What happens if $\Phi_{S_n} \longrightarrow \Phi_W$ as $n \to +\infty$

- **EXAMPLE**: $X_i \sim \text{Ber}\left(\frac{1}{n}\right)$; $i = 1, \dots, n$

  $\rightsquigarrow S_n \sim \text{Bin}\left(n, \frac{1}{n}\right)$,

  $$\Phi_{S_n}(t) = \left(\frac{1}{n} e^{it} + \left(1 - \frac{1}{n}\right)\right)^n$$

  $$\Phi_{S_n}(t) \xrightarrow[n \to \infty]{\text{PTWISE } \forall t} \exp\left[\lambda\left(e^{it} - 1\right)\right] = \Phi_W(t)$$

  for $W \sim P_o(\lambda)$.

• **THM**: (Lévy - Cramér Continuity Theorem)

(i): $F_n \Rightarrow F$ (weak convergence)

[That is: • $F_n(x) \xrightarrow{n\to\infty} F(x)$, $\forall$ x Cont.ty pt. of F]

[Equiv: • $\mathbb{P}_n([a,b]) \xrightarrow[n\to\infty]{} \mathbb{P}([a,b])$, $\forall a<b$: $\mathbb{P}(\{a\}) = \mathbb{P}(\{b\}) = 0$]

[Also denoted $X_n \xrightarrow{d} X$ for corresponding R.V.'s]

(ii): $\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\, d\mathbb{P}_n = \int_{\mathbb{R}} f(x)\, d\mathbb{P}$, $\forall f \in C_b(\mathbb{R})$

(iii): $\lim_{n\to\infty} \phi_n(t) = \phi(t)$, $\forall t \in \mathbb{R}$.

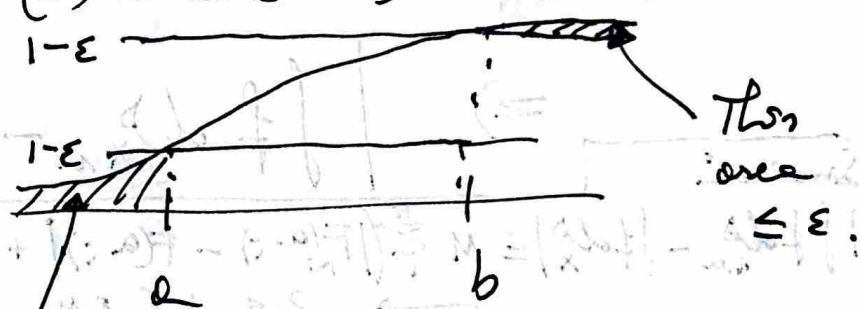
These statements (i), (ii), (iii) are EQUIVALENT.

PROOF :

(i) $\Rightarrow$ (ii) : f continuous & bdd.

First, restrict to $[a,b]$ finite interval s.t.

$F(a) \le \varepsilon$, $F(b) \ge 1-\varepsilon$



This area $\le \varepsilon$

$\rightsquigarrow$ lost in $\int f \, d\mu_x$ is at most $2\varepsilon M$

f uniformly cont's on $[a,b]$, choose $a = a_0 < a_1 < \dots < a_N = b$ continuity pts for $N(\varepsilon)$ s.t. $\sup_{a_i \le x,y \le a_{i+1}} |f(x) - f(y)| < \varepsilon$

$\rightsquigarrow$ ...

$$\left| \int f \, d\mu_{x_n} - \sum_1^N f(a_i) \left[F_n(a_i) - F_n(a_{i-1}) \right] \right|$$

$$\leq \varepsilon + C \cdot \varepsilon M$$

$$\left| \int f \, d\mu_{x_n} - \sum_1^N f(a_i) \left[F(a_i) - F(a_{i-1}) \right] \right| \leq \varepsilon + 4\varepsilon M$$

$$\Rightarrow \left| \int f \, d\mu_{x_n} - \int f \, d\mu_x \right| \leq 2\varepsilon + 2^* \varepsilon M.$$

Since:

$$\left| \int f \, d\mu_{x_n} - \int f \, d\mu_x \right| \leq M \sum_1^N \left(|F_n(a_i) - F(a_i)| + |F_n(a_{i-1}) - F(a_{i-1})| \right)$$

$$\xrightarrow{n \to \infty} \quad 2\varepsilon + C\varepsilon M \qquad \checkmark$$

(ii) $\Rightarrow$ (iii): $f_1(x) = \cos(tx)$; $f_2(x) = \sin(tx)$

bold & continuous, so

$$\lim_{n \to \infty} \int e^{itx} \, d\mu_{x_n} = \int_{\mathbb{R}} e^{itx} \, d\mu_x .$$

(iii) $\Rightarrow$ (i): (Sketch). Need

1. **Lemma:** (Helly - Bray) $\forall$ seq $(F_n)_n$ dist. on $\mathbb{R}$, $\exists$ G { RIGHT-CONT, non-decreasing on $\mathbb{R}$, and a subseq. (F_{n_k}) $0 \leq G \leq 1$. s.t.

s.t. $\lim_{n\to\infty} F_{n_n}(x) = G(x)$, $\forall$ x Cont. pt of G.

- diagonalization: $\exists\ G_0 : G_0(r_i) = \lim_{n\to\infty} F_{n_n}(r_i)$ (n_n)

where $\{r_i\}$ are the rationals.

define $G(x) = \inf\{G_0(r) : r > x,\ r \in \mathbb{Q}\}$.

(equiv: $= \lim_{r \downarrow x,\ r\in\mathbb{Q}} G_0(r)$)

END of LEMMA.

* G will be a dist. $\iff$ $G(-\infty) = 0$
$G(+\infty) = 1$.

(Unlike e.g. $F_n(x) = \begin{cases} 0, & x < n \\ 1, & x \geq n \end{cases}$

which $\to F_{x_n}$ for $x_n = n$
determinedly, $\lim_{n\to\infty} F_n(x) = G(x) \equiv 0$)

- If $\phi_{\mu_n}(t) \to \phi(t)$, $\forall t$ (iii) holds), then it must be that $\mathbb{P}_n$ is a tight sequence (mass does not run to ∞)

↳ $\forall \varepsilon > 0$, $\exists\, K > 0 : \mathbb{P}_n([-K, K]) > 1 - \varepsilon$, $\forall n$.

- Use this to conclude G must be a DISTR.

- Finally: $\begin{cases} F_{n_k} \Rightarrow G. \\ \phi_{\mu_{n_k}}(t) \to \phi(t) \end{cases}$,

then it must be that $\phi_{\mu} = \phi_G$.

This applies to ALL subseq of $(F_n)_n$, concluding the proof since:

$\exists\, x \in$ Cont. pt. of F s.t. $|F_{\tilde{n}_k}(x) - F(x)| \geq \delta$, $\forall k$, $\delta > 0$ but $\exists\, (\tilde{n}_{k_j}) : F_{\tilde{n}_{k_j}} \Rightarrow \tilde{G}$ at $\tilde{G} = F$, contradiction ■

W.L.L.N. (L^2 version)

$X_1, \dots, X_n$ uncorrelated ($Cov(X_i, X_j) = 0$, $\forall i \neq j$), $Var(X_i) \leq \sigma^2$, $\mathbb{E}(X_i) = \mu$ $\forall i$. Then,

$$\frac{1}{n} S_n \xrightarrow[n \to \infty]{L^2} \mu \qquad \left(S_n = \sum_1^n X_i \right)$$

PROOF:

$$\mathbb{E}\left| \frac{S_n}{n} - \mu \right|^2 = Var\left(\frac{S_n}{n}\right)$$

$$= \frac{1}{n^2} \sum Var(X_i) < \frac{\sigma^2}{n} \downarrow 0 \quad (n \to \infty)$$

(by Uncorr.) ■

* Is the assumpt. on $\mathbb{E}|X_i|^2$ needed for $\frac{S_n}{n} \xrightarrow{P} \mu$? → ...

• $\mathbb{E}|X_1| < \infty$

• **THM**: $X_1, \dots, X_n$ iid, $\mu := \mathbb{E} X_1 < \infty$, then

$$\frac{S_n}{n} \xrightarrow{L^1} \mu$$

PROOF: Truncation, fix $M > 0$.

$$X_i' = X_i \, \mathbb{I}_{\{|X_i| \le M\}},$$

$$Y_i = X_i - X_i' = X_i \, \mathbb{I}_{\{|X_i| > M\}}.$$

$$\mu' := \mathbb{E} X_i', \quad \eta := \mathbb{E} Y_i$$

$$\rightsquigarrow \mathbb{E}\left|\frac{S_n}{n} - \mu\right| \le \mathbb{E}\left|\frac{1}{n}\sum X_i' - \mu'\right| + \mathbb{E}\left|\frac{1}{n}\sum_1^n Y_i - \eta\right|$$

$$\le \left(\mathbb{E}\left|\frac{1}{n}\sum X_i' - \mu'\right|^2\right)^{\frac{1}{2}} + 2\,\mathbb{E}|Y_1|,$$

But $\frac{1}{n}\sum X_i' \xrightarrow{L^2} \mu'$, while —

$\mathbb{E}|y_1| \xrightarrow[M \uparrow \infty]{} 0$ (since

$\mathbb{E}|X_i| I_{\{|X_i| > M\}} \nearrow |X_i|$ as $M \uparrow \infty$

so $\mathbb{E}|X_i| I_{\{|X_n \le M\}} \to \mathbb{E}(X_i)$ as $M \uparrow \infty$

by $\underline{\underline{\text{MON}}}$).