PROBABILITY I, CLASS # 6

* **REMINDER**: Borel - Cantelli

- **LEMMA**: (B.C. I)

$$\left.\begin{array}{l} A_n \in \mathcal{F} \\ \sum_n \mathbb{P}(A_n) < \infty \end{array}\right\} \Rightarrow \mathbb{P}(\underbrace{A_n \text{ i.o.}}) = 0$$

$$= \overline{\lim} A_n = \bigcap_{n\geq 1} \bigcup_{m\geq n} A_m$$

[**PROOF:** $X(\omega) := \sum_1^\infty I_{A_n}(\omega)$, by MON

X is measurable and $\mathbb{E}X = \sum_1^\infty \mathbb{P}(A_n) < \infty$

but $\{A_n(\omega) \text{ i.o.}\} = \{X(\omega) = \infty\}$ so

$\mathbb{P}(A_n \text{ i.o.}) = 0$] ■

[✱: $\mathbb{I}_{A_n} = \mathbb{I}_{(0,\frac{1}{n}]}$ for $\omega \in [0,1]$]

↳ $\mathbb{P}(A_n \text{ i.o.}) = \lim_n \mathbb{P}(\bigcup_{m\geq n} A_m) = 0$,

(**Note** $\mathbb{P}(A_n) = \frac{1}{n} \Rightarrow \sum_1^\infty \mathbb{P}(A_n) = +\infty$.)

• **LEMMA** : (B.C. II)

Sort of converse under extra indep. assumption

If $A_n \in \mathcal{F}$ mutually independent & $\sum_n \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(A_n \text{ i.o.}) = 1$.

[PROOF: Fix $m < n < \infty$

$$\mathbb{P}\left(\bigcap_{m \le k \le n} A_k^c \right) = \prod_{k=m}^{n} \mathbb{P}(A_k^c)$$

$$\le e^{-\sum_{k=m}^{n} \mathbb{P}(A_k)}$$

$$\Rightarrow \mathbb{P}\left(\bigcap_{k \ge m} A_k^c \right) = \lim_{n \to \infty} \mathbb{P}\left(\bigcap_{k=m}^{n} A_k^c \right)$$

$$\le \lim_{n \to \infty} e^{-\sum_{k=m}^{n} \mathbb{P}(A_k)} = 0$$

$$\Rightarrow \mathbb{P}\left(\bigcup_{k \ge m} A_k \right) = 1, \ \forall m, \text{ so}$$

$$\mathbb{P}(A_n \text{ i.o.}) = 1 \ ] \blacksquare$$

* **RECALL:** Kolmogorov's Consistency (Extension) Thm gives rise to a unique distribution as the product measure on $\mathbb{R}^{\mathbb{N}}$ for prob. distributions on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$.

* **CONCLUSION:** SLLN under 4th moment assumpt:

$X_1, X_2, \ldots$ i.i.d. with $\mathbb{E}X_i^4 < K$, $\mathbb{E}X_i = 0$. Then,

$$\frac{S_n}{n} \xrightarrow[n\uparrow\infty]{\text{A.S.}} 0 \quad \text{for} \quad S_n = \sum_1^n X_i$$

PROOF: $\mathbb{E}(S_n)^4 = \sum_i \mathbb{E}X_i^4 +$

(Other terms $=0$ by indep. & $\mathbb{E}(X_i)=0$) $\longrightarrow$ $+ \sum_{i\neq j} \mathbb{E}(X_i^2)\,\mathbb{E}(X_j^2)$

$\leq nK + 3n(n-1)\cdot K$

$\leq 3n^2 K$ $\longrightarrow_{n\to\infty}$

$$\rightsquigarrow \mathbb{P}\left(\left|\frac{S_n}{n}\right| > \varepsilon\right) \leq \mathbb{P}\left(S_n^4 > (\varepsilon n)^4\right) \dots$$

$$\left(\text{Markov}\right) \longrightarrow \leq \frac{3n^2 K}{\varepsilon n^4}$$

$$\rightsquigarrow \sum_n \mathbb{P}\left(\left|\frac{S_n}{n}\right| > \varepsilon\right) < \sum_1^\infty \frac{C}{n^2} < \infty$$

$$\rightsquigarrow \text{by B. C. I, } \mathbb{P}\left(\left\{\left|\frac{S_n}{n}\right| > \varepsilon\right\} \text{ i.o.}\right) = 0$$

$$\therefore \frac{S_n}{n} \xrightarrow[n \uparrow \infty]{\text{A.S.}} 0 \quad \blacksquare .$$

* **LEMMA:** (Kolmogorov's Maximal Ineq.):

$X_1, \dots, X_n$ independent R.V.'s, $\mathbb{E} X_i = 0$, $\mathrm{Var}(X_i) < \infty$, $\forall i$. $S_n := \sum_1^n X_i$.

Then,

$$\mathbb{P}\left(\sup_{k \leq n} |S_k| \geq a\right) \leq \frac{\mathrm{Var}(S_n)}{a^2}$$

[Recall: From Chebycher's: $\mathbb{P}(|S_n| > a) \leq \frac{\mathrm{Var}(S_n)}{a^2}$
Kolgomorov's much stronger!]

$\rightsquigarrow \infty$

PROOF: $E_k = \{|S_{k-1}| \geq a\} \cap \bigcap_{i<k} \{|S_i| < a\}$

$$\left(\text{so } \{\max_{k \leq n} |S_k| \geq a\} = \biguplus_{k=1}^{n} E_k \right)$$

$$\rightsquigarrow \quad a^2 \, \mathbb{P}(E_k) \leq \mathbb{E}\left[S_k^2 \, \mathbb{I}_{E_k}\right]$$

$$\leq \mathbb{E}\left[\left(S_k^2 + (S_n - S_k)^2\right) \mathbb{I}_{E_k}\right]$$

$$= \mathbb{E}\left[\underbrace{\left(S_k^2 + (S_n - S_k)^2 + 2 S_k (S_n - S_k)\right)}_{= S_n^2} \mathbb{I}_{E_k}\right]$$

since $S_n - S_k \in \sigma(X_{k+1}, \ldots, X_n)$ and $\sigma(X_1, \ldots, X_k) \perp\!\!\!\perp \sigma(X_{k+1}, \ldots, X_n)$

so

$$\mathbb{E}(S_n - S_k) S_k \mathbb{I}_{E_k} = \mathbb{E}(S_n - S_k) \, \mathbb{E} S_k \mathbb{I}_{E_k} = 0$$

$$\Downarrow$$

$$\mathbb{P}(E_k) \leq \frac{\mathbb{E}\left[S_n^2 \mathbb{I}_{E_k}\right]}{a^2}$$

$$\therefore \quad \mathbb{P}\left(\biguplus_{1}^{n} E_k\right) = \sum_{1}^{n} \mathbb{P}(E_k) \leq \frac{\mathbb{E}(S_n^2)}{a^2} = \frac{\mathrm{Var}(S_n)}{a^2}$$

$$\left(\text{since } \sum_{1}^{n} \mathbb{I}_{E_k} \leq 1 \right) \qquad \blacksquare$$

* **THM**: (Kolmogorov's 1-Series Theorem)

$\{X_n\}_n$ indep. R.V.'s $\mathbb{E}X_n = 0$, $\mathrm{Var}(X_n) < \infty$, $\forall n$. If $\sum_n \mathrm{Var}(X_n) < \infty$, then $\sum_n X_n$ converges a.s.

(i.e., $S_n = \sum_1^n X_i$ satisfies $S_n \xrightarrow{a.s.} S_\infty$ which is finite).

PROOF: By Kolmogorov's Maximal Ineq:

$\forall \varepsilon > 0$, $n, m > 0$:

$$\mathbb{P}\left(\max_{n \le k \le n+m} |S_k - S_n| \ge \varepsilon \right) \le \frac{1}{\varepsilon^2} \sum_n^{n+m} \mathrm{Var}(X_i)$$

letting $m \uparrow \infty$:

$$\mathbb{P}\left(\sup_{k \ge n} |S_k - S_n| \ge \varepsilon \right) \le \frac{1}{\varepsilon^2} \sum_{i \ge n} \mathrm{Var}(X_i).$$

$\downarrow$ as $n \uparrow +\infty$ $\longrightarrow 0$

(Since $\sum_1^\infty \mathrm{Var}(X_i) < \infty$)

... So, if

$$R_n := \inf_{n,m \geq n} |S_n - S_m|,$$

Then $\forall \varepsilon > 0$:

$$\mathbb{P}\left(R_n \geq 2\varepsilon\right) \leq 2\mathbb{P}\left(\inf_{k \geq n} |S_k - S_n| \geq \varepsilon\right) \quad \text{(by } \Delta\text{-ineq.)}$$

$$\downarrow \; n \uparrow \infty$$

$$0$$

Since $n \longmapsto R_n(\omega)$ is non-increasing,

then

$$\mathbb{P}\left(\lim_{K \to \infty} R_K \geq 2\varepsilon\right) \leq \mathbb{P}\left(\inf_{k \geq n} R_k \geq 2\varepsilon\right)$$

$$\leq \mathbb{P}\left(R_n \geq 2\varepsilon\right)$$

$$\downarrow \text{as } n \uparrow \infty.$$

$$0$$

So $R_n(\omega) \xrightarrow[n \to +\infty]{\text{A.S.}} 0$: Thus, $S_n(\omega)$ is

a Cauchy A.S. ■ .

- **SLLN:** $\{X_n\}$ i.i.d., $\mathbb{E}|X_1| < \infty$, $\mathbb{E}X_1 = 0$, $S_n := \sum_1^n X_i$

  Then, $\frac{S_n}{n} \xrightarrow{a.s.} 0$.

- **CESÀRO'S LEMMA:** $\beta_n > 0$, $\beta_n \uparrow \infty$, $\alpha_n \to \alpha$, all in $\mathbb{R}$. Then,

$$\frac{1}{\beta_n} \sum_1^n (\beta_n - \beta_{n-1}) \alpha_n$$

$\downarrow n \uparrow \infty$

$$\alpha$$

PROOF:

$$\frac{1}{\beta_n} \sum_1^n (\beta_n - \beta_{n-1}) \alpha_n$$

$$\left(\begin{matrix} \alpha_n \geq \alpha - \varepsilon, \\ \forall n \geq N \end{matrix}\right) \longrightarrow \geq \underbrace{\frac{1}{\beta_n} \sum_1^N (\beta_n - \beta_{n-1}) \alpha_n}_{\xrightarrow{n \uparrow \infty} 0} + \frac{\beta_n - \beta_N}{\beta_n} (\alpha - \varepsilon)$$

$\downarrow n \uparrow \infty$

$$1$$

$$\longrightarrow \alpha - \varepsilon$$

. **KRONECKER'S LEMMA**: If $\beta_n > 0$, $\beta_n \uparrow \infty$,

and $\sum_n \frac{x_n}{\beta_n}$ converges, then

$$\frac{S_n}{\beta_n} \longrightarrow 0 \quad \left(\text{where } S_n = \sum_1^n x_i\right).$$

PROOF:

$$\left(\alpha_n := \sum_1^n \frac{x_k}{\beta_k} \; ; \; \alpha = \lim_{n\to\infty} \alpha_n \right),$$

$$\Rightarrow \quad \alpha_n - \alpha_{n-1} = \frac{x_n}{\beta_n} \qquad \infty$$

$$S_n = \sum_{k=1}^n \beta_k (\alpha_k - \alpha_{k-1}) = \ldots$$

$$\ldots = \beta_n \alpha_n - \sum_{k=1}^n (\beta_k - \beta_{k-1}) \alpha_{k-1}$$

$$\Rightarrow \frac{S_n}{\beta_n} = \alpha_n - \frac{1}{\beta_n} \sum_{k=1}^n (\beta_k - \beta_{k-1}) \alpha_{k-1}$$

$$\downarrow$$

$$\alpha - \alpha = 0$$

By Cesaro's ■

• Proof of SLLN:

let $y_n = X_n \mathbb{1}_{\{|X_n| \le n\}}$,

$$z_n = \frac{1}{n}\left(y_n - \mathbb{E}\, y_n\right)$$

⊛ Suppose we show that $\sum \mathrm{Var}(z_n) < \infty$.

Then, by Kolmogorov's 1-Series Thm, we will have that $\sum z_n$ converges A.S.

By B.C. I: $X_n = y_n$ eventually A.S.

since $\sum_n \mathbb{P}\left(X_n \neq y_n\right) = \sum_n \mathbb{P}\left(|X_n| > n\right)$

$$= \sum_n \mathbb{P}\left(|X_1| > n\right)$$

$z \ge 0$ a R.V., then
$\sum_n \mathbb{P}(z \ge n) \le \mathbb{E}\, z \le 1 + \sum_n \mathbb{P}(z \ge n)$ $\longrightarrow$ $\le \mathbb{E}|X_1| < \infty$

So $\sum_n \frac{1}{n}\left(X_n - \mathbb{E}\, y_n\right)$ converges A.S. as well.

So $\frac{1}{n}\sum_{k=1}^{n}\left(X_k - \mathbb{E}\, y_k\right) \xrightarrow{\text{A.S.}} 0$ by Kronecker's lemma,

$\longrightarrow$ done

while $\frac{1}{n}\sum_{k=1}^{n} \mathbb{E}(y_k) \longrightarrow 0$ by

Cesaro's, since $\mathbb{E}\, y_k \longrightarrow \mathbb{E}\, x_1$ by D.C.T.

(*): $\sum_n \mathrm{Var}(\beta_n) = \sum_n \frac{1}{n^2}\mathrm{Var}(y_n)$

$\leq \sum_n \frac{1}{n^2}\mathbb{E}|x_1|^2 \mathbb{1}_{\{|x_1|\leq n\}}$

$= \mathbb{E}(x_1^2)\sum_{n\geq |x_1|}\frac{1}{n^2}$

$\leq C\,\mathbb{E}|x_1| < \infty.$

(Since $\sum_{n\geq k}\frac{1}{n^2} \longrightarrow$

$\leq \sum_{n\geq k}\left(\frac{1}{n-1}-\frac{1}{n}\right)$

$\leq \frac{1}{k-1}-\frac{1}{L} \leq \frac{1}{k}$

where $k \simeq |x_1|$)