• **Class #7,** (Probability I).

• **Thm:** (Kolmogorov 2-Series Thm.)

$\{X_n\}$ indep., $\mu_n := \mathbb{E}(X_n)$, $\sigma_n^2 := \mathrm{Var}(X_n) < \infty$

If $\sum_1^\infty \mu_n < \infty$ and $\sum_1^\infty \sigma_n^2 < \infty$,

then $\sum_1^\infty X_n$ converges A.S..

PROOF: $\overline{X}_n := X_n - \mu_n$; by 1-Series Thm:

$\sum_1^\infty \overline{X}_n$ conv. A.S.,

while $\sum_1^\infty \mu_n < \infty$

$\rightsquigarrow \sum_1^\infty X_n$ conv A.S. ∎

- **Thm**: (Kolmogorov 3-Series Thm).

$\{X_n\}$ indep, then $\sum_1^\infty X_n$ converges A.S. IF & ONLY IF

$\exists\ c>0$ s.t.
$$\begin{cases} \sum_1^\infty \mathbb{P}(|X_n|>c) < \infty \\ \sum_1^\infty \mathbb{E}\left[X_n \mathbb{1}_{\{|X_n|\le c\}}\right] < \infty \\ \sum_1^\infty \mathrm{Var}\left(X_n \mathbb{1}_{\{|X_n|\le c\}}\right) < \infty \end{cases}$$

(*) Actually: $\sum_1^\infty X_n$ converges A.S.

$\Downarrow$

$\forall\, c>0$
$$\begin{cases} \sum_1^\infty \mathbb{P}(|X_n|>c) < \infty \\ \sum_1^\infty \mathbb{E}\left[X_n \mathbb{1}_{\{|X_n|\le c\}}\right] < \infty \\ \sum_1^\infty \mathrm{Var}\left(X_n \mathbb{1}_{\{|X_n|\le c\}}\right) < \infty \end{cases}$$

$\therefore$ Checking for $c=1$ suffices!!

PROOF:

($\Leftarrow$): $X_n' := X_n \mathbb{1}_{\{|X_n| \le c\}}$, c fixed.

$|X_n'| \le c$ by def, $\{X_n'\}$ indep,

$\sum_1^\infty \mathbb{E} X_n' < \infty$ and $\sum_1^\infty \mathrm{Var}(X_n') < \infty$

by assumption. By 2-Series

Thm, we know $\sum X_n'$ converges

A.S. Finally

$$\sum_n \mathbb{P}(X_n \neq X_n') = \sum_n \mathbb{P}(|X_n| > c) < \infty$$

by assumption again.

$\Rightarrow$ By. BC I, $\mathbb{P}(X_n \neq X_n' \text{ i.o.}) = 0$

Hence $\sum X_n$ and $\sum X_n'$ converge (or not) together.

Hence $\sum X_n$ conv $\Leftrightarrow$ $\sum X_n'$ conv.

on a set $E \subset \Omega$ with $\mathbb{P}(E) = 1$.

In particular, $\sum X_n$ conv. A.S.,

since we showed $\sum X_n'$ conv. A.S. ✓

($\Rightarrow$): Fix $c>0$.

- Since $\sum_1^\infty X_n < \infty$ A.S.,
we have $X_n \to 0$ A.S. $\Rightarrow$
$|X_n| \le c$ EVENTUALLY A.S. But $\{X_n\}$
are INDEP. $\Rightarrow$ by BC II we
must have

$$\sum_1^\infty \mathbb{P}(|X_n| > c) < \infty \quad \checkmark$$

- X_n' as above, by same argument
given above (BC I) we can
conclude $X_n = X_n'$ eventually A.S.
(just use $\sum_1^\infty \mathbb{P}(|X_n|>c) < \infty$, just showed)

$\Rightarrow \sum_1^\infty X_n'$ conv. A.S.

CLAIM: $\sum_1^\infty \mathrm{Var}(X_n') < \infty$ ([V], lemma 3.13 p. 47, slightly $\neq$ proof of this.)

This will conclude the proof
since:
For $\overline{X}_n' = X_n' - \mathbb{E}X_n'$ we have $\begin{cases} \mathbb{E}\overline{X}_n' = 0 \\ \sum_1^\infty \mathrm{Var}(\overline{X}_n') < \infty \end{cases}$ so...

∞ by 1-Series Thm: $\sum_1^\infty \overline{X^1}$ conv. A.S.

and since $\sum X_n^1$ conv. A.S. we

get $\sum \mathbb{E} X_n^1 < \infty$ ✓ (since $\mathbb{E} X_n^1 = X_n^1 - \overline{X_n^1}$)

- It remains to prove the claim:

Suppose $\sum_1^\infty \mathrm{Var}(X_n^1) = +\infty$. Then,

if $r_n = \sum_1^n \mathrm{Var}(X_i^1)$ we will see that

$$\hat{S}_n = \frac{1}{\sqrt{r_n}} \left(\sum_1^n \overline{X_i} \right)$$

satisfies $\hat{S}_n \xrightarrow{D} G \sim N(0,1)$

(Will follow from Lindeberg's CLT, since X_i^1 are indep. & bdd $\Rightarrow$ All moments exist...)

But: $\hat{S}_n = \frac{\sum_1^n X_i^1}{\sqrt{r_n}} - \frac{\sum_1^n \mathbb{E} X_i^1}{\sqrt{r_n}}$

We know $\sum_1^\infty X_i^1$ conv. A.S. so

$\frac{1}{\sqrt{r_n}} \sum_1^n X_i \xrightarrow[A.S.]{} 0$ (because $r_n \to +\infty$)

$\Rightarrow$ By Slutsky's lemma ("Converging together lemma") (Hw#4, Probl.#1),

$$\hat{S}_n + \frac{\sum_1^n X_i'}{\sqrt{v_n}} = \frac{\sum_1^n X_i'}{\sqrt{v_n}} \xrightarrow{D} G$$

$\hat{S}_n \downarrow$ a.s. 0 ; $\frac{\sum_1^n X_i'}{\sqrt{v_n}} \downarrow$ a.s. G

NON-RANDOM $\rightsquigarrow$ $\mathbb{P}\left(\frac{1}{\sqrt{v_n}} \sum_1^n X_i' > 0\right) \in \{0,1\}$,

whereas $\mathbb{P}(G > 0) = \frac{1}{2}$.

$\Rightarrow \Leftarrow$

- **Thm:** (Lindeberg CLT).

- **Warm up:** CLT for iid $X_1, \ldots, X_n$, $\mathbb{E}X_1 = 0$, $0 < \underbrace{\mathrm{Var}(X_1)}_{=:\sigma^2} < \infty$

Then, $\hat{S}_n = \frac{1}{\sqrt{n\sigma^2}} \sum_1^n X_i \xrightarrow{D} N(0,1)$.

Proof: $\infty\infty$

are $\Phi_{\hat{S}_n}(t) = \left(\Phi_{X_1}\left(\frac{t}{\sqrt{n}\sigma}\right)\right)^n$

Using $\Phi_{X_1}(t) = 1 - \frac{\sigma^2 t^2}{2} + o(t^2)$,

since $\Phi'_{X_1}(0) = \mathbb{E}X_1 = 0$ & $\Phi''_{X_1}(0) = \mathbb{E}X_1^2$,

gives $\Phi_{S_n}(t) = \left(1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n$

$\xrightarrow{n \uparrow \infty}$ $e^{-\frac{t^2}{2}} = \Phi_N(t)$,

where $N(0,1) \sim N$.

• **THM**: (Lindeberg's CLT).

Let $(X_{n,k})$ be a triangular array: 

For all $n = 1, 2, \ldots$ we have a probability space Ω_n and R.V.'s $X_{n,1}, \ldots, X_{n,n}$ which are $\mathbb{P}_n$-mutually independent (we assume this) s.t.

$$\mathbb{E}\hat{X}_{n,k} = 0$$

(Just NORMALIZE by multiplying $X_{n,k}$ by constant)

$$\hat{V}_n := \sum_{k=1}^{n} \mathbb{E}\hat{X}_{n,k}^2 \longrightarrow 1, \text{ as } n \to \infty.$$

$\mathbb{E}\hat{X}_{n,k}^2 = \mathrm{Var}(\hat{X}_{n,k}) = \mathrm{Var}(X_{n,k}) \cdot C(n)$

- Were $\hat{X}_{n,k}$ is just renormalized for $\hat{S}_n$ to have Variance 1
- Let $\hat{S}_n = \sum_{k=1}^{n} \hat{X}_{n,k}$

If: (Lindberg's Assumption)

$$\forall \varepsilon > 0, \quad g_n(\varepsilon) := \sum_{k=1}^{n} \mathbb{E}\left[\hat{X}_{n,k}^2 \mathbb{1}_{\{|\hat{X}_{n,k}| > \varepsilon\}}\right]$$

$$\downarrow n \uparrow \infty$$

$$0$$

LINDBERG's CONDITION (L.C.)

Then: $\hat{S}_n \xrightarrow{d} N(0,1)$.

- EXAMPLE: $X_1, \dots, X_n$ iid, $\sigma^2 = \mathrm{Var}(X_1) \in (0, \infty)$

$$\hat{X}_{n,k} = \frac{1}{\sigma\sqrt{n}} X_k$$

$$g_n(\varepsilon) = n\, \mathbb{E}\left[\hat{X}_{n,1}^2 \mathbb{1}_{\{|\hat{X}_{n,1}| > \varepsilon\}}\right]$$

$$= \frac{1}{\sigma^2} \mathbb{E}\left[X_1^2 \mathbb{1}_{\{|X_1| > \varepsilon\sigma\sqrt{n}\}}\right]$$

$$\xrightarrow[n \uparrow \infty]{} 0, \text{ by D.C.T.}$$