Class #8 : (PROBABILITY II)

* **Remark :** let $r_n^2 = \max\left\{\sigma_{n,k}^2 : k=1,\dots,n\right\}$

where $\sigma_{n,k}^2 = \mathbb{E}(\hat{X}_{n,k}^2)$

$$\boxed{\hat{V}_n = \sum_{k=1}^{n} \sigma_{n,k}^2} \xrightarrow{n\to\infty} 1$$

Then: $\forall\, k, \forall\, \varepsilon > 0$

$$\sigma_{n,k}^2 = \mathbb{E}\left[\hat{X}_{n,k}^2 \mathbb{1}_{\{|\hat{X}_{n,k}| < \varepsilon\}}\right] + \mathbb{E}\left[\hat{X}_{n,k}^2 \mathbb{1}_{\{|\hat{X}_{n,k}| \geq \varepsilon\}}\right]$$

$$\leq \varepsilon^2 + \beta_n(\varepsilon)$$

So: $r_n^2 \leq \varepsilon^2 + \beta_n(\varepsilon)$,

$\forall\, n, \forall\, \varepsilon > 0.$

$\Rightarrow$ (L.C.) implies that

$r_n \xrightarrow[n\uparrow\infty]{} 0$ (That is, the worst standard dev. $\sigma_{n,k}^2$ $(k=1,\dots,n) \longrightarrow 0$ as $n \uparrow +\infty$.

- **Feller:** $\hat{S}_n \xrightarrow{d} N(0,1)$, $r_n \xrightarrow[n\to\infty]{} 0$ $\Rightarrow$ [redacted] (L.C.)

(Lindeberg - Feller Thm.).

- **PROOF of Lindeberg's CLT:**

[redacted]

Idea: Replace $\hat{X}_{n,k}$ by indep. $Y_{n,k} \sim N(0, \sigma_{n,k}^2)$ iteratively, and control the effect of these replacements on $\mathbb{E}(h(\hat{S}_n))$ for a *smooth* h by Taylor expansion.

[We let $G_n = \sum_{k=1}^{n} Y_{n,k} \sim N(0, \hat{v}_n) = \sqrt{\hat{v}_n}\, N(0,1)$.
Note that we don't know distrib of $\hat{S}_n$, since $\hat{X}_{n,k}$ are k-indep. but possibly $\neq$ distrib & non-normal]

* **Key lemma:** let $h:\mathbb{R}\to\mathbb{R}$ be a function with continuous & uniformly bdd. 2nd & 3rd derivatives; let $G_n\sim N(0,\hat{v}_n)$

Then, $\forall n$, $\forall \varepsilon>0$,

$$\left|\mathbb{E}\,h(\hat{S}_n)-\mathbb{E}\,h(G_n)\right| \le \left(\frac{\varepsilon}{6}+\frac{r_n}{2}\right)\hat{v}_n\|h'''\|_\infty + g_n(\varepsilon)\|h''\|_\infty$$

- **Remark:** $G_n \overset{d}{=} \sqrt{\hat{v}_n}\,G$, for $G\sim N(0,1)$;

So, when $\hat{v}_n\to 1$ & $r_n\to 0$ (follows from C.L.), the lemma shows $\left|\mathbb{E}\,h(\hat{S}_n)-\mathbb{E}\,h(G_n)\right| =$

$$= \left|\mathbb{E}\,h(\hat{S}_n)-\mathbb{E}\,h\left(\sqrt{\hat{v}_n}G\right)\right| \xrightarrow{n\uparrow\infty} 0$$

Further,

$$\left|h\left(\sqrt{\hat{v}_n}x\right)-h(x)\right| \le \|h'\|_\infty\left|\sqrt{\hat{v}_n}-1\right||x|$$

So $\left|\mathbb{E}\,h\left(\sqrt{\hat{v}_n}G\right)-\mathbb{E}\,h(G)\right|\xrightarrow{n\uparrow\infty}0$,

provided h' is UNF. BDD. $\Longrightarrow\infty$

$\Rightarrow \lim_{n\to\infty} \mathbb{E}h(\hat{S}_n) = \mathbb{E}h(G)$,

$\forall h$ with continuous & uniformly bold first 3 derivatives.

(TECHNICAL LEMMA:)

* This will be enough to conclude $\hat{S}_n \xrightarrow{\mathcal{D}} N(0,1)$, because $\forall b$, there exists a sequence h_k^+, h_k^- $(k \geq 1)$ of functions with continuous & uniformly bold first 3 derivatives s.t.

$$\begin{cases} 0 \leq \bar{h}_k(x) \uparrow \mathbb{1}_{(-\infty,b)}(x) \\ \\ 1 \geqslant h_k^+(x) \downarrow \mathbb{1}_{(-\infty,b]}(x) \end{cases} \quad \text{as } k \uparrow +\infty$$

Then: (i): $\varlimsup_{n\to\infty} \mathbb{P}(\hat{S}_n < b) \geqslant \varlimsup_{n\to\infty} \mathbb{E}h_k^-(\hat{S}_n)$

$\left(h_k^-(x) \leq \mathbb{1}_{(-\infty,b)}(x)\right)$

$\left(\begin{matrix}\text{next}\\ \text{LEMMA}\end{matrix}\right) \longrightarrow = \mathbb{E}h_k^-(G) \uparrow \mathbb{P}(G<b)$ (by M.C.T.)

Similarly, $\left(h_k^+(x) \geq \mathbb{1}_{(-\infty, b]}(x)\right)$

(ii): $\overline{\lim_{n\to\infty}} \mathbb{P}(\hat{S}_n \leq b) \leq \overline{\lim_{n\to\infty}} \mathbb{E}\, h_n^+(\hat{S}_n)$

$\left(\begin{matrix}\text{KEY}\\ \text{LEMMA}\end{matrix}\right) \longrightarrow = \mathbb{E}\, h_n^+(G)$

$\left(\begin{matrix}\text{by}\\ \text{B.C.T.}\end{matrix}\right) \rightarrow \downarrow \mathbb{P}(G \leq b)$ ∎

- **left to PROVE KEY LEMMA:**

Let $G_n = \sum_{k=1}^{n} y_k$ where $y_k \sim N(0, \sigma_{n,k}^2)$.

Write X_k as short for $\hat{X}_{n,k}$, so that

$$\hat{S}_n = \sum_{k=1}^{n} X_k$$

Define $U_\ell = \sum_{k=1}^{\ell-1} X_k + \sum_{k=\ell+1}^{n} y_k \quad (\ell = 1, \ldots, n-1)$

U_ℓ: $\quad X_1\ X_2\ \cdots\ X_{\ell-1}\ \boxed{X_\ell}\ X_{\ell+1}\ \cdots\ X_n$

$\quad\quad y_1\ y_2\ \cdots\ y_{\ell-1}\ \boxed{y_\ell}\ y_{\ell+1}\ \cdots\ y_n$

(Ignore ℓ^{th} elements)

Then, $G_n = u_1 + y_1$

$$\hat{S}_n = u_n + x_n$$

$$u_\ell + x_\ell = u_{\ell+1} + y_{\ell+1}, \quad (\ell = 1, \dots, n-1)$$

So

$$\left| \mathbb{E}\, h(G_n) - \mathbb{E}\, h(\hat{S}_n) \right| =$$

$$= \left| \mathbb{E}\, h(u_1 + y_1) - \mathbb{E}\, h(u_n + x_n) \right|$$

$$\leq \sum_{\ell=1}^{n} \Delta_\ell \quad , \quad \text{where}$$

$$\Delta_\ell = \left| \mathbb{E}\left[h(u_\ell + y_\ell) - h(u_\ell + x_\ell) \right] \right| \longrightarrow \cdots$$

We did this:

$$\underbrace{h(u_1 + y_1) - h(u_1 + x_1)}_{\leq \Delta_1} + \underbrace{h(u_1 + x_1)}_{\| \; h(u_2 + y_2)} - h(u_n + x_n)$$

... and continue up to $n-1$ ..., (& use Δ-inequality)

∞∞ let

$$R_e(\xi) := h(u_e+\xi) - h(u_e) - \xi h'(u_e) - \frac{\xi^2}{2} h''(u_e)$$

Then,

(**)
$$\begin{cases} |R_e(\xi)| \le \|h'''\|_\infty \dfrac{|\xi|^3}{6} & \left(\begin{matrix}3^{rd}\text{ order}\\ \text{Taylor}\end{matrix}\right) \\ |R_e(\xi)| \le \|h''\|_\infty |\xi|^2 & \left(\begin{matrix}2^{nd}\text{ order}\\ \text{Taylor}\end{matrix}\right) \end{cases}$$

← (no extra $\frac{1}{2}$ because of extra $\frac{\xi^2}{2}$ term in def. of $R_e(\xi)$).

and

$$h(u_e + x_e) = h(u_e) + x_e h'(u_e) + \frac{x_e^2}{2} h''(u_e) + R_e(x_e)$$

$$h(u_e + y_e) = h(u_e) + y_e h'(u_e) + \frac{y_e^2}{2} h''(u_e) + R_e(y_e)$$

∞∞

... So by subtracting 2 terms :

$$\Delta_e \le \left| \mathbb{E}\left[(X_e - Y_e) h'(U_e)\right] \right| +$$

$$+ \left| \mathbb{E}\left[\tfrac{1}{2}(X_e^2 - Y_e^2) h''(U_e)\right] \right|$$

$$+ \left| \mathbb{E}\left[R_e(X_e) - R_e(Y_e)\right] \right|$$

But $\quad \mathbb{E} X_e = \mathbb{E} Y_e$

$\mathbb{E} X_e^2 = \mathbb{E} Y_e^2$

& $U_e \perp\!\!\!\perp \sigma(X_e, Y_e)$ ($X_{n,e}$ are indep. within rows, i.e. for n fixed & Y_e's are by assumpt indep.)

so $\Delta_e \le \mathbb{E}\left|R_e(X_e)\right| + \mathbb{E}\left|R_e(Y_e)\right|$

By (**) :

$$\# \left| R_e(X_e) \right| \le \|h'''\|_\infty \mathbb{E}\left[\frac{|X_e|^3}{6} \perp\!\!\!\perp_{\{|X_e| \le \varepsilon\}}\right] \quad (3^{\text{rd}} \text{ order})$$

$$+ \|h''\|_\infty \mathbb{E}\left[|X_e|^2 \perp\!\!\!\perp_{\{|X_e| \ge \varepsilon\}}\right] \quad (2^{\text{nd}} \text{ order})$$

$$\le \infty$$

$$\infty \le \frac{\varepsilon}{6} \|h'''\|_\infty \mathbb{E}(X_e^2)$$

$$+ \|h''\|_\infty \mathbb{E}\left[X_e^2 \mathbb{1}_{\{|X_e| \ge \varepsilon\}}\right]$$

So

$$\sum_{e=1}^{m} \mathbb{E}|R_e(X_e)| \le \frac{\varepsilon}{6}\|h'''\|_\infty V_m + \|h''\|_\infty \varphi_m^{(\varepsilon)}$$

Finally, $\frac{Y_e}{\sigma_{m,e}} \sim N(0,1)$, so

$$\mathbb{E}\left[\left(\frac{Y_e}{\sigma_{m,e}}\right)^4\right] = 3, \quad \mathbb{E}\left[\left|\frac{Y_e}{\sigma_{m,e}}\right|^3\right] \le 3$$

$$\Rightarrow \mathbb{E}|Y_e|^3 \le 3\,\sigma_{m,e}^3 \le 3\, r_m\, \sigma_{m,e}^2$$

$\Rightarrow$ by (**):

$$\sum_{e=1}^{m} \mathbb{E}|R_e(Y_e)| \le \frac{1}{6}\|h'''\|_\infty \sum_{e=1}^{m} \mathbb{E}|Y_e|^3$$

$$\le \frac{1}{2}\|h'''\|_\infty\, r_m V_m$$

■