• Class # 9 — PROBABILITY I

• REMARK : Lyapunov's Condition:

$S_n = \sum_{k=1}^{n} X_k$ independent R.V.'s

$\sigma_n^2 := \mathrm{Var}(S_n)$. If, for some $q > 2$, we have

$$\frac{1}{\sigma_n^q} \sum_{k=1}^{n} \mathbb{E}|X_k - \mathbb{E}X_k|^q \xrightarrow[n \to \infty]{} 0$$

then Lindeberg's condition (*) holds.

(N.B. for $q = 2$ the LHS is 1:

$$\frac{1}{\mathrm{Var}(S_n)} \cdot \mathrm{Var}(S_n)$$

• CONDITIONAL EXPECTATION:

- let $X, Z \in SF$ on $(\Omega, \mathcal{F}, \mathbb{P})$

$X \in \{x_1, \dots, x_m\}$, $Z \in \{z_1, \dots, z_n\}$

$\mathbb{P}(X = x_i) > 0$, $\mathbb{P}(Z = z_j) > 0$

From basic probability:

$$\mathbb{P}(X = x_i \mid Z = z_j) = \frac{\mathbb{P}(X = x_i, Z = z_j)}{\mathbb{P}(Z = z_j)}$$

(For fixed j: a prob. distrib.)

$$\mathbb{E}[X \mid Z = z_j] = \sum_i x_i \, \mathbb{P}(X = x_i \mid Z = z_j)$$

- That is: let $y(\omega)$ be the R.V. denoted $\mathbb{E}[X \mid Z]$ equal to $\mathbb{E}[X \mid Z = z_j]$ for $z_j = Z(\omega)$. → ...

$\Rightarrow$ (ii): $y(\omega)$ is constant on each

$$G_J = \{\omega : \ z(\omega) = z_J\}$$

(iii): $y(\omega)$ is measurable on $\mathcal{G} = \sigma(Z)$

$$= \{ Z^{-1}(B) : B \in \mathcal{B}_R \}$$

(here: $\mathcal{G}$ is the 2^n subsets of $\sigma\{z_1, \dots, z_n\}$ — pre-images of)

(iii): If $G \in \mathcal{G}$, then $G = \biguplus_{J \in \mathcal{J}} G_J$ for $\mathcal{J} \subseteq \{1, \dots, n\}$ and so

$$\mathbb{E}[y \mathbb{1}_G] = \sum_{J \in \mathcal{J}} \mathbb{E}[y \mathbb{1}_{G_J}]$$

$$= \sum_{J \in \mathcal{J}} \mathbb{E}[X / Z = z_J] \cdot \mathbb{P}(Z = z_J)$$

$$= \sum_{J \in \mathcal{J}} \sum_{i=1}^{n} x_i \, \mathbb{P}(X = x_i, Z = z_J)$$

$$= \mathbb{E}[X \mathbb{1}_G].$$

(This was for simple functions).

* **THM**: (Conditional Expectation)

Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-field.

~~Then: There exists $Y \in L^1(\Omega, \mathcal{G}, \mathbb{P})$~~

(1): $\exists$ a $\mathcal{G}$-measurable R.V. Y denoted $\mathbb{E}[X|\mathcal{G}]$ (a version of) s.t. $Y \in L^1(\Omega, \mathcal{G}; \mathbb{P})$, and

(*): $\mathbb{E}[Y \mathbb{1}_G] = \mathbb{E}[X \mathbb{1}_G]$, $\forall G \in \mathcal{G}$

(2): If (*) holds for Y and Y' in $L^1(\Omega, \mathcal{G}, \mathbb{P})$ Then $Y = Y'$ $\mathbb{P}$-a.s.

. We write $\mathbb{E}[X|Z]$ to denote $\mathbb{E}[X|\sigma(Z)]$ and $\mathbb{E}[X|Z_1, Z_2, \ldots] = \mathbb{E}[X|\sigma(Z_1, Z_2, \ldots)]$

• PROOF of UNIQUENESS (2):

Let $G_\delta = \{\omega : y(\omega) - y'(\omega) > \delta\}$, $\delta > 0$.

Since $G_\delta \in \mathcal{G}$ we have by (*)

$$\mathbb{E}[X \mathbb{1}_{G_\delta}] = \mathbb{E}[y \mathbb{1}_{G_\delta}]$$

$$\mathbb{E}[X \mathbb{1}_{G_\delta}] = \mathbb{E}[y' \mathbb{1}_{G_\delta}]$$

so $$0 = \mathbb{E}[(y - y') \mathbb{1}_{G_\delta}] > \delta \, \mathbb{P}(G_\delta)$$

Take $\delta \downarrow 0$ ($\leadsto G_\delta \uparrow G_0 = \{y > y'\}$)

$$\leadsto \mathbb{P}(y > y') = 0 \quad \text{(by cont. from below of } \mathbb{P})$$

by symmetry: $\mathbb{P}(y' > y) = 0$ ■

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* EXERCISE: [D 4.1.3]: $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra. If $y \in L^1(\Omega, \mathcal{G}, \mathbb{P})$ has $\mathbb{E}[X \mathbb{1}_G] = \mathbb{E}[y \mathbb{1}_G]$ $\forall\, G \in \mathcal{D}$ for some π-system $\mathcal{D}$ s.t. $\Omega \in \mathcal{D}$ and $\mathcal{G} = \sigma(\mathcal{D})$. Then $y = \mathbb{E}[X \mid \mathcal{G}]$

Sol: ...

∞ Need to show $\mathbb{E} \times \mathbb{1}_G = \mathbb{E}\,y\,\mathbb{1}_G$ $\forall\, G \in \mathcal{G} = \sigma(\mathcal{P})$. Write

$$\mathcal{L} = \{ A \in \mathcal{F} : \mathbb{E} \times \mathbb{1}_A = \mathbb{E}\,y\,\mathbb{1}_A \},$$

then by hypothesis: $\mathcal{P} \subseteq \mathcal{L}$.

Suffices to show $\mathcal{L}$ is a λ system (so by Dynkin's π-λ Theorem $\sigma(\mathcal{P}) = \mathcal{G} \subseteq \mathcal{L}$)

Indeed:
- $\Omega \in \mathcal{P} \subseteq \mathcal{L}$ so $\Omega \in \mathcal{L}$
- If $A, B \in \mathcal{L}$ then $B \setminus A \in \mathcal{L}$ $A \subseteq B$ (just subtract integrals)
- If $A_i \in \mathcal{L}$, $A_i \uparrow A$ then $A \in \mathcal{L}$ by D.C.T.

□

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* <u>PROOF of EXISTENCE:</u> (Part ①)

- <u>Recall:</u> If μ, ν are measures on $(\Omega, \mathcal{F})$ a measurable space, we say $\nu \ll \mu$ (ν is ABS. CONT.'s w.r.t. μ) if $\mu(A) = 0 \Rightarrow \nu(A) = 0$, $\forall A \in \mathcal{F}$.

- Recall: $\forall\, f \in m\mathcal{F}_+$ (measurable ≥ 0) induces on the measure space $(S, \mathcal{F}, \mu)$ a measure $f\mu$ given by

$$(f\mu)(A) = \int_A f \, d\mu$$

$$= \mathbb{E}_\mu[f \mathbb{1}_A], \quad \forall A \in \mathcal{F}.$$

(Indeed,

$f\mu \geq 0$ because $f \geq 0$

$f\mu(\emptyset) = 0$

If $A_n \uparrow A$, Then

$0 \leq f \mathbb{1}_{A_n} \uparrow f \mathbb{1}_A$ as $n \uparrow \infty$.

so

$$\mathbb{E}_\mu[f \mathbb{1}_A] \overset{\text{M.C.T.}}{=} \lim_{n\to\infty} \mathbb{E}_\mu[f \mathbb{1}_{A_n}]$$

$$\infty \mathbb{E}_\mu[f \mathbb{1}_{A_n}]$$

so $(f\mu)$ is continuous from below & finitely add. $\Rightarrow f\mu$ is a measure. ∎

* REMINDER (3):

Thm (Radon-Nikodym):

- If μ, ν are σ-finite measures on $(\Omega, \mathcal{F})$ and $\nu << \mu$ then $\exists f \in m\mathcal{F}_+$, finite valued s.t. $\nu = f\mu$.

  (f denoted as $\frac{d\nu}{d\mu}$ the RADON NIKODYM DERIVATIVE (RND)).

- Further, if $f\mu = g\mu \Rightarrow f(\omega) = g(\omega)$ for μ - a.e. ω.

* To show the existence of C.E.:

First take $X \geq 0$.

$\mu = \mathbb{P}$ restricted to $\mathcal{G}$

$\nu(A) = (X\mu)(A) = \int_A X \, d\mu$, where

- ν is a FINITE MEASURE: $\nu(\Omega) = \mathbb{E}X < \infty$.
- If $\mu(G) = 0$, then $\nu(G) = 0 \Rightarrow \nu << \mu$

∞

By R-N Thm: $\exists$ R.V. $y \in m\mathcal{G}_+$ s.t. $\nu = y\mu$. That is,

$\forall G \in \mathcal{G}$: $\mathbb{E}[X \mathbb{1}_G] = \nu(G) = \mathbb{E}_\mu[y \mathbb{1}_G]$.

For general X: Write $X = X^+ - X^-$

let $y^+ = \mathbb{E}[X^+ | \mathcal{G}]$

$y^- = \mathbb{E}[X^- | \mathcal{G}]$

Then $y = y^+ - y^-$ satisfies

$\mathbb{E}[y \mathbb{1}_G] = \mathbb{E}[y^+ \mathbb{1}_G] - \mathbb{E}[y^- \mathbb{1}_G]$

$= \mathbb{E}[X^+ \mathbb{1}_G] - \mathbb{E}[X^- \mathbb{1}_G] = \mathbb{E}[X \mathbb{1}_G]$.

* WARNING: In general, we <u>won't</u> expect to have $\mathbb{E}[X|\mathcal{G}]^{\pm} = \mathbb{E}[X^{\pm}|\mathcal{G}]$

That is, don't expect to have in general

$\mathbb{E}[X|\mathcal{G}]^+ = \mathbb{E}[X^+|\mathcal{G}]$

or $\mathbb{E}[X|\mathcal{G}]^- = \mathbb{E}[X^-|\mathcal{G}]$

E.g.: $X = \begin{cases} 2 & , \frac{1}{2} \\ -2 & , \frac{1}{2} \end{cases}$

$\mathcal{G} = \{\emptyset, \Omega\}$

$Y := \mathbb{E}[X|\mathcal{G}] = 0$

$\mathbb{E}[X^+|\mathcal{G}] = 1$

$\mathbb{E}[X^-|\mathcal{G}] = 1.$