CLASS # 10 PROBABILITY II

* PROPERTIES of C.E.

([V] Thm. 4.5
[D] Thm 5.6 4.2)

let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $Y = \mathbb{E}[X|\mathcal{G}]$ for $\mathcal{G} \subseteq \mathcal{F}$

(i): $\mathbb{E}Y = \mathbb{E}X$

$\left(\mathcal{G} \supset \Omega = G : \quad \mathbb{E}X = \mathbb{E}X\mathbb{1}_\Omega = \mathbb{E}Y\mathbb{1}_\Omega = \mathbb{E}Y\right)$

(ii): $X \geq 0 \Rightarrow Y \geq 0$

$\left(G = \{\omega : Y(\omega) \leq 0\}, \quad G \in \mathcal{G}\right.$

$0 \leq \mathbb{E}[X\mathbb{1}_G] = \mathbb{E}[Y\mathbb{1}_G] \leq 0$

$\Rightarrow \mathbb{E}[Y\mathbb{1}_G] = 0$

$\Rightarrow Y \geq 0 \quad A.S.$

Moreover $\mathbb{P}(X>0, Y \leq 0) = 0 \left(\underset{A.S.}{X>0 \Rightarrow Y>0}\right)$

$\longrightarrow$

$\Rightarrow$ (because $\forall \delta > 0$:

$$\delta \, \mathbb{P}(X > \delta, Y \leq 0) \leq \mathbb{E}[X \mathbb{1}_{\{X>\delta\}} \mathbb{1}_{\underbrace{G}_{Y \leq 0}}]$$

$$\leq \mathbb{E}[X \mathbb{1}_G] = 0).$$

(iii): $\mathbb{E}[\alpha X_1 + \beta X_2 | \mathcal{G}] = \alpha \mathbb{E}[X_1|\mathcal{G}] + \beta \mathbb{E}[X_2|\mathcal{G}]$

$\in L^1(\Omega, \mathcal{F}, \mathbb{P})$

(iv), (v): <u>Jensen's:</u> $\varphi(x)$ is convex on $\mathcal{O} \in \mathbb{R}$ open interval s.t. $X \in \mathcal{O}$ and $X, \varphi(X) \in L^1$, then:

$$\mathbb{E}[\varphi(X)|\mathcal{G}] \geq \varphi(Y) \quad \text{a.s.}$$

and taking expectation:

$$\mathbb{E}[\varphi(X)] \geq \mathbb{E}[\varphi(Y)]$$

(V): Tower Property:

$\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$: $\mathbb{E}[X|\mathcal{H}] = \mathbb{E}[\underbrace{\mathbb{E}[X|\mathcal{G}]}_{=Z}|\mathcal{H}]$

$\left(Y = \mathbb{E}[X|\mathcal{G}] \in L^1 \Rightarrow Z = \mathbb{E}[Y|\mathcal{H}] \in L^1 \right)$

$\forall\, H \in \mathcal{H}$, $H \in \mathcal{G}$ hence

$$\mathbb{E}[Y \mathbb{1}_H] = \mathbb{E}[Z \mathbb{1}_H]$$

also $\mathbb{E}[X \mathbb{1}_H] = \mathbb{E}[Y \mathbb{1}_H]$

* [D 4.2.3]:

If $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{H} \perp\!\!\!\perp \sigma(\mathcal{G}, \sigma(X))$

then $\mathbb{E}[X|\sigma(\mathcal{G},\mathcal{H})] = \mathbb{E}[X|\mathcal{G}]$

• COROLLARY: $(\mathcal{G} = \{\emptyset, \Omega\})$

If $\mathcal{H} \perp\!\!\!\perp \sigma(X)$ then $\mathbb{E}[X|\mathcal{H}] = \mathbb{E}[X]$

PROOF: $\forall\, G \in \mathcal{G}$, $H \in \mathcal{H}$: $\mathbb{1}_H \perp\!\!\!\perp \sigma(X\mathbb{1}_G, Y\mathbb{1}_G)$

so $\mathbb{E}[X \underbrace{\mathbb{1}_{G\cap H}}_{=\mathbb{1}_G\mathbb{1}_H}] = \underbrace{\mathbb{E}[X\mathbb{1}_G]}_{\overset{(C.8)}{=}} \mathbb{E}\mathbb{1}_H$

$\mathbb{E}[Y\mathbb{1}_G]\,\mathbb{E}\mathbb{1}_H = \mathbb{E}[Y\mathbb{1}_{G\cap H}]$

... We showed $\mathbb{E}[X\mathbb{1}_A] = \mathbb{E}[Y\mathbb{1}_A]$

for all $A \in \mathcal{D} = \{G \cap H : G \in \mathcal{G}, H \in \mathcal{H}\}$

but $\mathcal{D}$ is a π-system:

$$\Omega \in \mathcal{D}, \quad \sigma(\mathcal{D}) = \sigma(\mathcal{G}, \mathcal{H}),$$

& by [D 4.1.3] the result follows. ∎

* If $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $Z \in m\mathcal{G}$, $XZ \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ then $\mathbb{E}[XZ|\mathcal{G}] = Z\,\mathbb{E}[X|\mathcal{G}]$.

(Prop 4.2.10 [D] : proved with the std machine).

* C.E. as an orthogonal projection:

Consider $u, v \in L^2(\Omega, \mathcal{F}, \mathbb{P})$

$$\langle u, v \rangle := \mathbb{E}[uv]$$

and $\|u\|_2^2 = \langle u, u \rangle$

- $\|u+v\|_2^2 = \|u\|_2^2 + \|v\|_2^2$ if $\langle u, v \rangle = 0$

  ↓

  write $u \perp v$

- $\|u+v\|_2^2 + \|u-v\|_2^2 = \langle u+v, u+v \rangle + \langle u-v, u-v \rangle$

  $= 2\|u\|_2^2 + 2\|v\|_2^2$

NOTE: $\|u\|_2 = 0 \iff u \overset{\text{a.s.}}{=} 0$

- L^2 is a COMPLETE NORMED VECTOR SPACE

  (X_n Cauchy in L^2, then $X_n \xrightarrow[L^2]{n \uparrow \infty} X \in L^2$).

PROOF:

let $k_N \uparrow \infty$ s.t. $\sup_{m,n \geq k_N} \| X_m - X_n \|_2 \leq 2^{-N}$

$$\mathbb{E} \left| X_{k_{N+1}} - X_{k_N} \right| \underset{\text{C.S.}}{\leq} \| X_{k_{N+1}} - X_{k_N} \|_2 \leq 2^{-N}$$

$\Downarrow$

$$\mathbb{E} \sum \left| X_{k_{N+1}} - X_{k_N} \right| < \infty$$

$$\Rightarrow \sum \left(X_{k_{N+1}} - X_{k_N} \right) \text{ conv. A.S.}$$

$$\Rightarrow \exists \lim_{N \to \infty} X_{k_N}(\omega) \quad \text{A.E. } \omega \in \Omega$$

Set

$$X(\omega) = \overline{\lim_{N \to \infty}} X_{k_N}(\omega) \qquad \left(\text{So } X \in m\mathcal{F} \quad X_{k_N} \xrightarrow{A.S.} X \right)$$

$\forall N \in \mathbb{N}$, $n \geq k_N$, $t \geq N$:

$$\mathbb{E} \left| X_n - X_{k_N} \right|^2 = \| X_n - X_{k_t} \|_2^2 \leq \leq 2^{-2N}$$

$\rightsquigarrow$ ∞

∞ Taking $t \to \infty$ + Fatou:

$$\|X_n - X\|_2^2 \le 2^{-2N} \quad \text{so} \quad X_n \xrightarrow{L^2} X$$

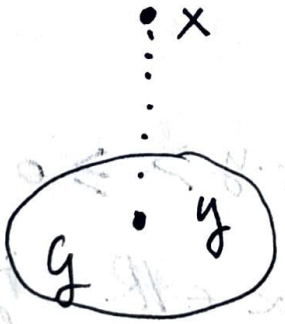
* $\boxed{\text{Thm:}}$ If $K \subseteq L^2$ complete vector sub-space, then $\forall\ X \in L^2\ \exists\ Y \in K$ s.t.

(i): $\|X - Y\|_2 = \inf\{\|X - W\|_2 : W \in K\}$.

(ii): $X - Y \perp Z$, $\forall\ Z \in K$

(i) $\Leftrightarrow$ (ii) and Y attaining then Y is A.S. UNIQ

(if Y, Y' attain, say (i), then $Y \overset{A.S.}{=} Y'$)



PROOF: let (Y_n) s.t. $\|X - Y_n\|_2 \longrightarrow \Delta$

$$\underbrace{\|X - Y_m\|_2^2}_{\downarrow\ \Delta^2} + \underbrace{\|X - Y_n\|_2^2}_{\downarrow\ \Delta^2} = 2\underbrace{\left\|X - \overbrace{\frac{Y_m + Y_n}{2}}^{\in K}\right\|_2^2}_{\ge \Delta^2} + 2\left\|\frac{Y_m - Y_n}{2}\right\|_2^2$$

$$\Rightarrow \quad \| y_m - y_n \|_2 \longrightarrow 0$$

by hyp: $\exists\, y : \| y_n - y \|_2 \longrightarrow 0$

Note: $\| x - y \|_2 \leq \| x - y_n \|_2 + \| y_n - y \|_2$

$\| y_n - y \|_2 \to 0$

$\Rightarrow \| x - y \|_2 = 0$ (if not, $\Rightarrow\Leftarrow$).

This establishes (i). ✓

For (ii): If $z \in K$, then $\forall\, \alpha \in \mathbb{R}$:

$$\| x - (y + \alpha z) \|_2^2 \geq \| x - y \|_2^2$$

$\Rightarrow$

$$\alpha^2 \| z \|_2^2 - 2\alpha \langle z, x - y \rangle \geq 0$$

This can only hold for $\alpha \in \mathbb{R}$ if

$$\langle z, x - y \rangle = 0. \quad \checkmark$$

■

* **C.E. for $X \in L^1$:** (EXISTENCE WITHOUT RAD.-NYK. Thm.)

Suffices to take $X \geq 0$ (otherwise $X = X^+ - X^-$ etc...)

Take $X_n \geq 0$, bounded, $X_n \uparrow X$.

Each X_n is in L^2: it has $y_n = \mathbb{E}[X_n | \mathcal{G}]$.

Since $0 \leq X_n$ and $\uparrow$, we have $0 \leq y_n \uparrow \infty$ (here we used that $\mathbb{E}(X_n|\mathcal{G}) \geq 0$ if $X_n \geq 0$).

Let $y(\omega) := \overline{\lim_{n \to \infty}} y_n(\omega)$

So $y \in m\mathcal{G}$, $y_n \uparrow y$ a.s.

By M.C.T.: $\mathbb{E}[y \mathbb{1}_G] = \mathbb{E}[X \mathbb{1}_G]$, $\forall G \in \mathcal{G}$

thanks to the fact $\mathbb{E}[y_n \mathbb{1}_G] = \mathbb{E}[X_n \mathbb{1}_G]$, $\forall n$.

■

* **Thm.** : (C.E. as an orth. projection)

If $\mathbb{E}X^2 < \infty$ then the C.E. $Y = \mathbb{E}[X|\mathcal{G}]$ is a version of the ORTHOGONAL PROJ. of X onto $L^2(\Omega, \mathcal{G}, \mathbb{P})$.

PROOF: For $\mathcal{G} \subseteq \mathcal{F}$, let $\mathcal{K} = L^2(\Omega, \mathcal{G}, \mathbb{P})$:

we showed that it is complete

$\exists\ y \in \mathcal{K}$ s.t.

(i): $\mathbb{E}[(X-y)^2] = \inf\left\{\mathbb{E}[(X-w)^2] : w \in L^2(\Omega, \mathcal{G}, \mathbb{P})\right\}$

(ii): $\mathbb{E}[(X-y)z] = 0$, $\forall\ z \in L^2(\Omega, \mathcal{G}, \mathbb{P})$

Take $G \in \mathcal{G}$; the R.V. $z = \mathbb{1}_G \in L^2(\Omega, \mathcal{G}, \mathbb{P})$

has $\mathbb{E}[\underbrace{yz}_{\mathbb{1}_G}] = \mathbb{E}[\underbrace{Xz}_{\mathbb{1}_G}]$

so y is a version of $\mathbb{E}[X|\mathcal{G}]$.