- **CLASS #11** : (PROBABILITY II)

- Suppose $(X,Z) \in \mathbb{R}^2$ has joint density $f_{X,Z}(x,z)$. Then,

$$\begin{cases} f_Z(z) := \int_{\mathbb{R}} f_{X,Z}(x,z)\, dx & (\text{i.e., } \# f_{X,Z}(\cdot,z) \text{ w.r.t. } \mathcal{L}\text{-measure}) \\ f_X(x) := \int_{\mathbb{R}} f_{X,Z}(x,z)\, dz \end{cases}$$

both non-negative, $\mathcal{B}$-measurable fn's.

So

$$f_{X|Z}(x|z) := \begin{cases} \dfrac{f_{X,Z}(x,z)}{f_Z(z)}, & f_Z(z) > 0 \\ f_X(x), & f_Z(z) = 0 \end{cases}$$

satisfies $f_{X|Z} : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ Borel fn;

$$\int_{\mathbb{R}} f_{X|Z}(x|z)\, dx = 1, \quad \forall z \in \mathbb{R}.$$

- **Proposition ([D., 4.4.1])**:

$\forall\, g : \mathbb{R} \to \mathbb{R}$ Borel s.t. $\mathbb{E}|g(X)| < \infty$, then $\hat{g}(z)$ is a version of $\mathbb{E}[g(X)|Z]$ where

$$\hat{g}(z) = \begin{cases} \int_{\mathbb{R}} g(x) f_{X|Z}(x|z)\, dx, & \text{if } < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(proved via def. of $f_{X|Z}$ and use Fubini, and to justify Fubini by Jensen's ineq.)

* $f_{X|Z}(\cdot|\cdot)$ gives rise to ~~$P_{X|Z}(B,\omega) = \int_B f_{X|Z}$~~

$$\hat{P}_{X|Z}(B,\omega) = \int_B f_{X|Z}(x|z(\omega))\, dx$$

(take above : $g(x) = \mathbb{1}_{\{x \in B\}}$).

* Generalization:

- **DEF**: (Regular conditional probability Distrib.)

Let $Y: \Omega \to \mathcal{D}$ be an $(\mathcal{D}, \mathcal{Y})$-valued R.V. on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-field. We call $\hat{\mathbb{P}}_{Y|\mathcal{G}}$ the REGULAR CONDITIONAL PROB. DISTRIB of Y given $\mathcal{G}$ if:

(a): $\hat{\mathbb{P}}_{Y|\mathcal{G}}(B, \cdot)$ is a version of $\mathbb{E}[\mathbb{1}_{\{Y \in B\}} | \mathcal{G}]$, $\forall B \in \mathcal{Y}$

(b): $\hat{\mathbb{P}}_{Y|\mathcal{G}}(\cdot, \omega)$ is a probability measure on $(\mathcal{D}, \mathcal{Y})$ $\forall \omega \in \Omega$.

(By default: Take $\mathcal{D} = \Omega$, $\mathcal{Y} = \mathcal{F}$, $Y(\omega) = \omega$. $\rightsquigarrow$ this is the R.C.P. of $\mathbb{P}$ given $\mathcal{G}$: $\hat{\mathbb{P}}(B|\mathcal{G})(\omega)$)

* Might not exist!

• **PROPOSITION**: ([D. 4.4.3.])

$\forall X$ REAL VALUED R.V. and $\forall \mathcal{G} \subseteq \mathcal{F}$,

$\exists$ R.C.P.D. $\hat{\mathbb{P}}_{X|\mathcal{G}}(\cdot,\cdot)$

[Idea: $\sigma(X)$ inherits the structure of the $\mathcal{B}$ σ-field which is "not too BIG" since $\mathbb{Q}$ dense in $\mathbb{R}$.

In general: can have

$$\hat{\mathbb{P}}\left(\bigcup_n B_n \,|\, \mathcal{G}\right)(\omega) \neq \sum_n \hat{\mathbb{P}}(B_n|\mathcal{G})(\omega)$$

for $\omega \in B_0$ with $\mathbb{P}(B_0) = 0$

(see Thm 4.7 in [V.])].

[• [D 4.4.4] More generally, whenever $(S, \mathcal{T})$ is isomorphic to $(\mathbb{R}, \mathcal{B})$ then last proposition holds.]

* **MARKOV CHAINS**:

- A <u>filtration</u> $\mathcal{F}_n$ is a non-decreasing seq. of sub-σ-fields: $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_\infty$. $\sigma\left(\bigcup_n \mathcal{F}_n\right)$
- A <u>Stochastic Process</u> X_n <u>adapted to</u> $\mathcal{F}_n$ is a seq. of R.V.'s s.t.

$$\sigma(X_n) \subseteq \mathcal{F}_n \quad , \forall n.$$

* **DEF**: (M.C.)

An $\mathcal{F}_n$-adapted stoch. process (X_n), which is $(S, \mathcal{J})$-valued, is a <u>Markov Chain</u> if $\forall A \in \mathcal{J}$: $\mathbb{P}(X_{n+1} \in A \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} \in A \mid X_n)$ a.s., $\forall n$.

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* <u>By default</u>: take the "canonical" filtration

$$\mathcal{F}_n := \mathcal{F}_n^X = \sigma(X_k : k \le n).$$

(This is also a M.C. if X_n is an $\mathcal{F}_n$-adapted M.C. $\rightsquigarrow$ ...

$$\cdots \quad \mathbb{P}\left(X_{n+1} \in A \,/\, \mathcal{F}_n^X\right) =$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{X_{n+1}\in A} \,/\, \mathcal{F}_n\right] \,/\, \mathcal{F}_n^X\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{X_{n+1}\in A} \,/\, X_n\right] \,/\, \mathcal{F}_n^X\right] = \mathbb{E}\left[\mathbb{1}_{X_{n+1}\in A} \,/\, X_n\right]$$

(X_n is $\mathcal{F}_n$ adapted M.C.)

$$= \mathbb{P}\left(X_{n+1} \in A \,/\, X_n\right)$$

(Were $\mathcal{F}_n^X \subseteq \mathcal{F}_n$ by assumption).

* **DEF.** : (transition probabilities)

$p : S \times \mathcal{S} \longrightarrow [0,1]$ is a transition probability if (S: CURRENT STATE of chain; $\mathcal{S}$: where chain goes given current state).

(a) : $A \longmapsto p(x,A)$ is a probability measure on $(S,\mathcal{S})$ $\forall x \in S$

(b) : $x \longmapsto p(x,A)$ is a meas. function on $(S,\mathcal{S})$ $\forall A \in \mathcal{S}$. (ie, for fixed A, $p(\cdot,A)$ is a R.V. on $(S,\mathcal{S})$).

* An $\mathcal{F}_n$ - M.C. (X_n) is said to have trans. probab. $p_n(\cdot,\cdot)$ if

$$\mathbb{P}(X_{n+1} \in A \mid \mathcal{F}_n) = p_n(X_n, A), \quad \forall n \geq 0, \quad \forall A \in \mathcal{S}.$$

* (X_n) is <u>HOMOGENEOUS</u> if $p_n(\cdot,\cdot) \equiv p(\cdot,\cdot)$ $\forall n$.
(no dependence on time for the trans. prob.)

* **PROPOSITION ([D 6.1.5])** : ✱ ⊛△

$\forall$ $p_n(\cdot,\cdot)$ - transition prob., $\forall$ ν σ-finite measure on $(\mathcal{S}, \mathcal{S})$ (← initial distrib), $\exists$ <u>unique</u> σ-finite measures, for $n = 0, 1, \ldots$

$$\mu_n = \nu \otimes p_0 \otimes \cdots \otimes p_{n-1} \text{ on } (\mathcal{S}^{n+1}, \mathcal{S}^{n+1})$$

s.t., $\forall A_i \in \mathcal{S}$, $i = 0, \ldots, n$

$$\mu_n(A_0 \times A_1 \times \ldots \times A_n) = \int_{A_0} \nu(dx_0) \int_{A_1} p_0(x_0, dx_1) \cdots \cdots \int_{A_n} p_{n-1}(x_{n-1}, dx_n)$$

And if (X_n) is an $\mathcal{F}_n$-adapted MC with transition probab. $p_n(\cdot,\cdot)$ and initial distr. ν, then $\forall n \geq 0$, $\forall h_i \in m\mathcal{B}$ bounded,

$$\mathbb{E}\left[\prod_0^n h_i(X_i)\right] = \int \nu(dx_0) h_0(x_0) \int p_0(x_0, dx_1) h(x_1) \cdots \int p_{n-1}(x_{n-1}, dx_n) h_n(x_n)$$

In particular: if $h_i = \mathbb{1}_{A_i}$, then

$$\mathbb{P}(X_0 \in A_0, \ldots, X_n \in A_n) = (\nu \otimes p_0 \otimes \cdots \otimes p_{n-1})(A_0 \times \cdots \times A_n)$$

* R.C.P.D.: $\forall$ $\mathcal{F}_n$-MC with a $\mathcal{B}$-isomorphic state space have transition prob.'s (From now on assumed).

* The Markov Property: (Strong).

- A R.V. $\tau : \Omega \longrightarrow \{0, 1, 2, \ldots\}$ is a STOPPING TIME for a filtration $(\mathcal{F}_n)$ if

$$\{\omega : \tau(\omega) \leq n\} \in \mathcal{F}_n \quad \forall \text{ fixed } n \geq 0.$$

• **EXAMPLE:** $\tau_A := \min\{t : X_t \in A\}$ is an $\mathcal{F}_n^X$ - stopping time

(first visit, or hitting time for A;
also: second visit, but NOT LAST VISIT)

• **NOTATION:** θ shift operator on S^∞,

$\theta^n(\omega_0, \omega_1, \ldots) = (\omega_n, \omega_{n+1}, \ldots)$.

$\mathcal{F}_C$ = σ-field associated to S^∞, generated by cylinder sets

$b\mathcal{F}_C$ = set of bounded measurable functions on $\mathcal{F}_C$.

• **PROPOSITION** : (Markov Property ; Strong M.P.)

(a): (X_n) homogeneous M.C. started at ν, $h_n \in b\mathcal{F}_c$ $\forall n$, $\sup_{n,\omega} |h_n(\omega)| < \infty$, then $\forall n \geq 0$

$$\mathbb{E}_\nu \left[h(\theta^n \omega) \,|\, \mathcal{F}_n^X \right] = \mathbb{E}_{X_n} [h] \quad \text{(M.P.)}$$

[Meaning :

$\mathbb{E}$ of $h(\omega_n, \omega_{n+1}, \dots)$ given $\sigma(X_0^N, X_1, \dots, X_n)$ is same as

$\mathbb{E}$ of $h(\tilde{X}_0, \tilde{X}_1, \dots)$ with $\tilde{X}_0 = X_n$]

(b): $h_n \in b\mathcal{F}_c$ $\forall n$, $\sup_{n,\omega} |h_n(\omega)| < \infty$, and

τ an $\mathcal{F}_n^X$-stopping time, then

$$\mathbb{E}_\nu \left[h_\tau (\theta^\tau \omega) \,|\, \mathcal{F}_\tau^X \right] \mathbb{1}_{\{\tau < \infty\}} = \mathbb{E}_{X_\tau} [h_\tau] \, \mathbb{1}_{\{\tau < \infty\}} \quad \text{(S.M.P.)}$$

[Replaces fixed n by random stopping time !]

(Stopped σ-field : $\mathcal{F}_\tau = \{ A \in \mathcal{F}_\infty : A \cap \{\tau \leq n\} \in \mathcal{F}_n, \forall n \}$
"everything known when we condition on τ".)