- **Class #12**: (PROBABILITY I)

- Take-home Final Exam, 2 days time

- Stopped σ-field $\mathcal{F}_\tau$ for an $\mathcal{F}_n$-stopping time τ: [See [D] 5.1.35]

$$\mathcal{F}_\tau := \{A \in \mathcal{F}_\infty : A \cap \{\omega : \tau(\omega) \le n\} \in \mathcal{F}_n, \forall n\}$$

(where $\mathcal{F}_\infty = \bigvee_0^\infty \mathcal{F}_n$). This satisfies:

(i): $\mathcal{F}_\tau$ is a σ-field. (e.g.: $\Omega \in \mathcal{F}_\tau$ by def. of stopp. time. etc ...).

(ii): If $\tau(\omega) = n$ A.S., then $\mathcal{F}_\tau = \mathcal{F}_n$

(iii): If $X_n \in m\mathcal{F}_n$ (including $n=\infty$), then $X_\tau \in m\mathcal{F}_\tau$ (eg: consider $A = \{X_\tau \in B\}$; show also that $A \in \mathcal{F}_\infty$:
$\{X_\tau \in B\} \cap \{\tau \le n\} = \bigcup_{\ell=0}^{n} \{X_\tau \in B\} \cap \{\tau = \ell\}$
$= \bigcup_{\ell=0}^{n} \{X_\ell \in B\} \cap \{\tau = \ell\} \in \mathcal{F}_n$.)

In particular, $\sigma(\tau) \subset \mathcal{F}_\tau$

and $X_k \mathbb{1}_{\{\tau=k\}} \in m\mathcal{F}_\tau$

for non-random k.

($X_\tau \mathbb{1}_{\{\tau=k\}} \in m\mathcal{F}_\tau$ $\forall$ non-random k

...)

(IV): For k non-random

$$\mathbb{E}[Y_\tau \mathbb{1}_{\{\tau=k\}} | \mathcal{F}_\tau] =$$

$$= \mathbb{E}[Y_k | \mathcal{F}_k] \mathbb{1}_{\{\tau=k\}}$$

(use def. of c.e. to prove this).

(V): If $\theta \le \tau$ are both $\mathcal{F}_n$-stopping times:

$\mathcal{F}_\theta \subseteq \mathcal{F}_\tau$.

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* Markov Property: Follows from ⊛ △ In Class # 11, eg:

$$\mathbb{E}_\nu[h(\theta^m w) \mathbb{1}_B(w_0, \dots, w_m)] = \mathbb{E}_\nu[\mathbb{1}_B(X_0, \dots, X_m) \mathbb{E}_{X_m}[h]] \quad \infty$$

∞ if $h = \prod_{0}^{\kappa} f_i(\omega_0)$.

Thus extends to any $h \in b\mathcal{F}_c$, where $\mathcal{F}_c$ = cylinders σ-field using MCT. (see [D., p. 233]). ■

* Strong Markov Property :

def. $g(n, x_n) = \mathbb{E}_\nu [h_n(\theta^n \omega) | \mathcal{F}_n^x]$

$\mathbb{E}[h_\tau(\theta^\tau \omega) \mathbb{1}_{\tau<\infty} | \mathcal{F}_\tau] =$

(since τ is $\mathcal{F}_\tau$-meas.)

$= \sum_\kappa \mathbb{E}[h_\tau(\theta^\tau \omega) \mathbb{1}_{\tau=\kappa} | \mathcal{F}_\tau]$

$= \sum g(\kappa, x_\kappa) \mathbb{1}_{\{\tau=\kappa\}}$ ← by part (iii) of previous Ex.

$= g(\tau, x_\tau) \mathbb{1}_{\tau=\kappa}$

■

* Countable MC's: (usually homogeneous)

$P(\cdot,\cdot)$: countably-inf. stochastic matrix.

let $\tau_x = \inf\{t \geq 1 : X_t = x\}$

$\tau_A = \inf\{t \geq 1 : X_t \in A\}$

• **Def:** $\rho(x,y) = \mathbb{P}_x(\tau_y < \infty)$

(i): y is "ACCESSIBLE" from x if $\rho(x,y) > 0$

(ii): $x \leftrightarrow y$ "INTER-COMMUNICATE" if $\begin{cases} \rho(x,y) > 0 \\ \rho(y,x) > 0 \end{cases}$ or $x = y$.

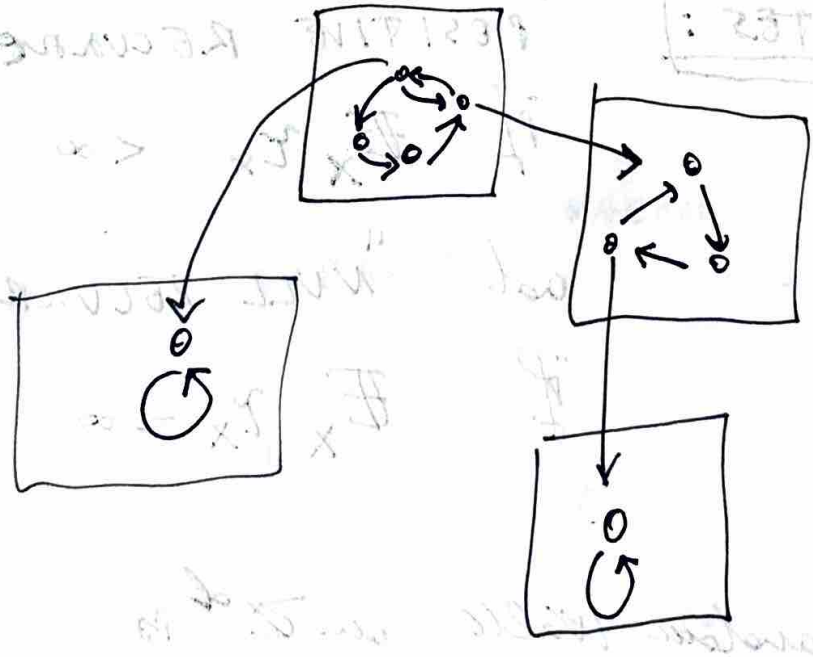
~~[illegible]~~

(iii): $S \subset \mathcal{S}$ is "IRREDUCIBLE" if (countable) $x \leftrightarrow y, \forall x \neq y \in S$.

"CLOSED" if $\nexists\, x \in S, y \notin S$: $\rho(x,y) > 0$.

- **NOTE**: $\leftrightarrow$ is an equiv. relation; its equivalence classes correspond to the "strongly connected" components of the directed graph on Ω where (x,y) is an edge iff $\rho(x,y) > 0$

* An equivalence class is exactly the maximal irreducible subsets of Ω.



* **DEF**: $x \in S$ is "RECURRENT" if

$$\underbrace{P(x,x) = 1}.$$

$$\mathbb{P}_x\left(\inf\{t \geq 1 : X_t = x\} < \infty\right) = \mathbb{P}_x(\tau_x < \infty) = 1$$

and it is "TRANSIENT" otherwise:

$$\mathbb{P}_x(\tau_x = \infty)$$

* **RECURRENT STATES:**

"POSITIVE RECURRENT" if $\mathbb{E}_x \tau_x < \infty$

and "NULL RECURRENT" if $\mathbb{E}_x \tau_x = \infty$

EX: Simple Random Walk on $\mathbb{Z}^d$ is recurrent for $d = 1, 2$ and transient for $d \geq 3$.

- **OBS:**

* If $p(x,y) > 0$ so $\Rightarrow \mathbb{P}_x(\tau_y < \tau_x)$.

(if $\mathbb{P}_x(\tau_y < \tau_x) = 0$ then X_n will visit x ∞-often before visiting y, so it will never visit y A.S.:

$\forall\, k \geq 1$: $\mathbb{P}_x(\tau_y = k) = \sum_{j=1}^{k-1} \mathbb{P}_x(\tau_x = j)\, \underbrace{\mathbb{P}_x(\tau_y = k-j)}_{=0 \text{ by induct.}}$

$+ \underbrace{\mathbb{P}_x(\tau_y < \tau_x)}_{=0 \text{ by ASSMPT.}}$

STRONG MARKOV PROPERTY:

$= 0$

$X_{\tau_x} = x$.

$\mathbb{P}_x(\tau_x < \tau_y, \tau_y = k)$

∞

So sum the $\mathbb{P}_x(\tau_y = k)$ over k to get contradiction.)

* If $\mathcal{S}$ is IRREDUCIBLE then $\forall$ state is of same type.

* **OBS:**

If $\tau_y^{(k)} = \inf\{t \geq k : X_1, \ldots, X_t \text{ visits } y \text{ } k \text{ times}\}$

$=$ "time of k-th visit to y",

then $\mathbb{P}_x(\tau_y^{(k)} < \infty) = p(x,y)\, p(y,y)^{k-1}$.

Again: Use Strong Markov Property on $\tau_y^{(k-1)}$ inductively:

$$\mathbb{P}_x(\tau_y^{(k)} < \infty) = \mathbb{E}_x\left[\mathbb{E}_x\left[\mathbb{1}_{\{\tau_y^n < \infty\}} \mid \mathcal{F}_{\tau_y^{(k-1)}}\right] \mathbb{1}_{\tau_y^{(k-1)} < \infty}\right]$$

$$\left(\begin{matrix}\text{STRONG}\\ \text{MARKOV}\end{matrix}\right) \longrightarrow = \mathbb{E}_x\left[p(y,y)\, \mathbb{1}_{\{\tau_y^{(k-1)} < \infty\}}\right]$$

$$= p(y,y)\, \mathbb{P}_x(\tau_y^{(k-1)} < \infty)$$

& induction ... ∎

* **Conclusion:** $N_\infty(y) := \sum_{n=1}^{\infty} \mathbb{1}_{\{X_n = y\}}$

satisfies:

$$G(x,y) := \mathbb{E}_x N_\infty(y) = \frac{\rho(x,y)}{1-\rho(y,y)}$$

$$\sum_{n=1}^{\infty} \mathbb{P}_x\left(N_\infty(y) \geq n\right) = \sum_{1}^{\infty} \mathbb{P}_x\left(\tau_y^{(n)} < \infty\right)$$

$$= \sum_{1}^{\infty} \rho(x,y)\,\rho(y,y)^{n-1}$$

■

* **RECAP:** y is "RECURRENT" if $\rho(y,y) = 1$

$$\iff \mathbb{P}_y\left(\tau_y^n < \infty\right) = 1, \ \forall\, n.$$

$$\iff \mathbb{P}_y\left(X_n = y \text{ i.o.}\right) = 1$$

$$\iff \mathbb{P}_y\left(N_\infty(y) = \infty\right) = 1$$

$$\iff G(y,y) = \mathbb{E}_y N_\infty(y) = \infty.$$

■

* $\mathcal{S}$ COUNTABLE IS UNIQUELY DECOMPOSED into

$$T \oplus R_1 \oplus R_2 \oplus \cdots$$

