- **CONSTRUCTION OF MEASURES**:

- **THM**: (Caratheodory Extension)

Any countably additive probability measure P on an algebra $\mathcal{A}$ extends uniquely as a countably additive probability measure to the σ-field $\mathcal{F}$ generated by $\mathcal{A}$.

(Uniqueness follows from Dynkin's π-λ theorem: P a π system (closed under finite $\cap$'s), $\mathcal{L}$ a λ-system ($\Omega \in \mathcal{L}$; $A, B \in \mathcal{L}, A \subseteq B \Rightarrow B \setminus A \in \mathcal{L}$; $\mathcal{L}$ closed under increasing limits), then $P \subseteq \mathcal{L}$ $\Rightarrow$ $\sigma(P) \subseteq \mathcal{L}$).

- Extension theorem does not yet solve problem of constructing countably additive p. measures. It reduces it to constructing these on an algebra A.

- Construction on $\mathbb{R}$: Take Borel σ-algebra

$$\mathcal{B}_{\mathbb{R}} = \sigma\left(\{\text{Open sets in } \mathbb{R}\}\right) = \sigma\left(\{(a,b] : -\infty \le a < b \le +\infty\}\right)$$

Consider class of subsets.

$$\mathcal{I} = \{(a,b] : -\infty \le a < b \le +\infty\}$$

The class of sets that are finite disjoint unions of members of $\mathcal{I}$ is an algebra A, if ϕ is added to it.

Take $F(x)$ on $\mathbb{R}$ which is non-decreas., right continuous & $F(-\infty)=0$, $F(+\infty)=1$

Define finitely additive measure P:

$$P((a,b]) = F(b) - F(a)$$

∞ →

for members of $\mathcal{I}$, then extend to algebra $\mathcal{A}$ by defining it as the union for disj. unions from $\mathcal{I}$.

NOTE: $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A})$

THM: P is COUNTABLY ADDITIVE on $\mathcal{A} \iff F(x)$ is a RIGHT CONT's function.

Therefore, for each right cont's non-decreasing function with $F(-\infty)=0$, $F(+\infty)=1$, $\exists!$ probability measure P on $\mathcal{B}_{\mathbb{R}}$ s.t. $F(x) = P((-\infty, x])$.

Conversely, every countably additive prob. measure P on $\mathcal{B}_{\mathbb{R}}$ comes from some F. ($P \leftrightarrow F$ is one-to-one).

In other words, we defined a finitely additive measure P on algebra $\mathcal{A}$ and by Extension Thm., extended it! to $\sigma(\mathcal{A}) = \mathcal{B}$ by showing P is countably additive on $\mathcal{A}$.

- Given any pr. measure P on $\mathcal{B}_{\mathbb{R}}$, let $F(x) = P((-\infty, x])$. By properties of pr. measure, F is right cont's, non-decreasing & $F(-\infty) = 0$, $F(+\infty) = 1$.

- The function F is called the DISTRIBUTION FUNCTION corresponding to the probability measure F.

Ex: If $f(x) \geq 0$ integrable fn with $\int_{-\infty}^{+\infty} f(s)\,ds = 1$, then $F(x) := \int_{-\infty}^{x} f(s)\,ds$ is a DISTRIB. FUNCTION which is continuous. f is the DENSITY of the measure P_F and can be calculated as $f(x) = F'(x)$.

- Consider $X:(\Omega_1, \mathcal{F}_1) \longrightarrow (\Omega_2, \mathcal{F}_2)$ a R.V., P a probability measure on $(\Omega_1, \mathcal{F}_1)$ and $Q := PX^{-1}$ the induced p.measure on $(\Omega_2, \mathcal{F}_2)$.

- **THM**: If $f:(\Omega_2, \mathcal{F}_2) \longrightarrow (\mathbb{R}, \mathcal{B})$ is measurable, then $g(\omega_1) := f(X(\omega_1))$ is measurable & g is integrable w.r.t. P $\Leftrightarrow$ f is integrable w.r.t. Q, and

$$\int_{\Omega_2} f(\omega_2)\, dQ = \int_{\Omega_1} g(\omega_1)\, dP$$

(Were both sides may be ∞)

- If $X:(\Omega_1, \mathcal{F}_1) \longrightarrow (\Omega_2 = \mathbb{R}, \mathcal{F}_2 = \mathcal{B})$, then for $f(x) = x$, we have

$$\int_{\Omega_1} X(\omega_1)\, dP = \int_{\mathbb{R}} x\, dQ$$

- The induced measure $\underline{Q = PX^{-1}}$ is called the DISTRIBUTION of R.V. X under P.

- From now on we use R.V. to describe measurable map $X: (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B})$.

  ↳ It induces a PROBABILITY DISTRIBUTION α on $\mathcal{B}$ by $\alpha := P X^{-1}$. The distrib. function corresponding to α is

$$F(x) := \alpha((-\infty, x]) = P[\omega : X(\omega) \le x].$$

(α = "DISTRIBUTION of X", $F(x)$ = "DISTRIB. FUNCTION of X".)

- If $g: (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ measurable, then $Y(\omega) := g(X(\omega))$ is again a R.V. with distribution $\beta = PY^{-1} = PX^{-1}g^{-1} = \alpha g^{-1}$.

→

∞ By Transformation theorem:

$$\mathbb{E}(g(X)) = \int_{\Omega} g(X(\omega))\, dP$$

$$= \int_{\mathbb{R}} g(x)\, d\alpha$$

(Same for RANDOM VECTORS $X=(X_1,\dots,X_n)$ and joint distrib. of X given by $\alpha = P X^{-1}$ on $\mathbb{R}^n$).

• RIEMANN - STIELTJES INTEGRAL

The R-S integral uses the distribution functions directly to define $\int_{-\infty}^{+\infty} g(x)\, dF(x)$, where $g \in C_b(\mathbb{R})$ and F a distrib. fn. It is defined as limit as $N \to \infty$ of sums

$$\sum_{j=0}^{N} g(x_j)\left[F(a_{j+1}^N) - F(a_j^N)\right]$$

where $-\infty < a_1^N < a_2^N < \dots < a_{N+1}^N < +\infty$ is a partition of the finite interval $[a_0^N, a_{N+1}^N]$ and the limit is taken in such a way that $a_0^N \to -\infty$, $a_{N+1}^N \to +\infty$, and the oscillation of g in any $[a_j^N, a_{j+1}^N]$ goes to zero. Show that if P is the measure corresponding to F, then

$$\int_{-\infty}^{+\infty} g(x)\, dF(x) = \int_{\mathbb{R}} g(x)\, dP .$$

* PROOF: • Fix $\varepsilon > 0$, $\delta > 0$ arbitrary.

• Take $a, b \in \mathbb{R}$ s.t. (a, b are cont. pts of F &) $F(a) \le \varepsilon$, $1 - F(b) \le \varepsilon$ (possible since $F(-\infty) = 0, F(+\infty) = 1$)

• Find a partition of $[a, b]$ by dividing $[a, b]$ into $N = N_\delta$ (finite) subintervals $I_J = [a_J^N, a_{J+1}^N]$ s.t. all endpts $\{a_J^N\}_{J=1}^{N+1}$:

$$a = a_1^N < a_2^N < \dots < a_{N+1}^N = b,$$

(possible since # of disc. pts of F are countable), and s.t. the oscillation of g on each I_J is less than δ (possible since g is UNIFORMLY continuous on $[a, b]$ so just chose N_δ sufficiently large).

Take simple function $h(x) := \sum_{J=1}^{N} g(x_J) I_{I_J}$.

Then, for $x \in [a, b]$, $|g(x) - h(x)| < \delta$.

($x_J \in (a_J^N, a_{J+1}^N]$ arbitrary). → cont'd

$\infty\infty$ Thus,

$$\left| \int_{\mathbb{R}} g(x)\, dP - \sum_{1}^{N} g(x_j) \left[F(a_{j+1}^{N}) - F(a_j^{N}) \right] \right| =$$

$$= \left| \int_{(a,b]} (g(x) - h(x))\, dP + \int_{\mathbb{R}\setminus(a,b]} g(x)\, dP \right|$$

$$\leq \delta \cdot 1 + M\, P(\mathbb{R}\setminus(a,b])$$

$$= \delta + M \left[1 - F(b) + F(a) \right]$$

$$\leq \delta + 2\varepsilon M.$$ Since $\varepsilon, \delta > 0$ are arbitrary, conclude that

$$\int_{-\infty}^{+\infty} g(x)\, dF(x) = \int_{\mathbb{R}} g(x)\, dP. \quad \blacksquare$$

• COMPUTING $\mathbb{E}$ using $f_X(x)$:

Suppose X a R.V., $X:(\Omega,\mathcal{F},\mathbb{P})\longrightarrow(\mathbb{R},\mathcal{B})$ with law $\mathcal{P}_X(B)=\mathbb{P}(X^{-1}(B))$, $B\in\mathcal{B}$ and distrib. function $F_X(b)=\mathcal{P}_X((-\infty,b])=\dots$ $\dots=\int_{-\infty}^{b} f_X(x)\,dx$, ($f_X\geq 0$, f_X integrates to 1 on $\mathbb{R}$, f $\mathcal{L}$-integrable...). Then, if $h:\mathbb{R}\to\mathbb{R}$ a $\mathcal{B}$-measurable function, then

$$\mathbb{E}(h(X))=\int_{-\infty}^{+\infty} h(x)\,f_X(x)\,dx.$$

PROOF: Note that $\mathbb{E}(h(X))=\int_{\mathbb{R}} h(x)\,d\mathcal{P}_X$, thus if $h(x)=I_{(-\infty,b]}\Rightarrow \mathbb{E}(h(X))=\mathcal{P}_X(-\infty,b]=F_X(b)=\dots$ $\dots=\int_{-\infty}^{b} f_X(x)\,dx=\int_{-\infty}^{+\infty} h(x)\,f_X(x)\,dx$. Further, $\mu(B):=\int_{-\infty}^{+\infty} I_B(x)f_X(x)\,dx$ is a probab. measure ...

... on $(\mathbb{R}, \mathcal{B})$ (easy to show...).

Thus, we showed $\mathcal{P}_X = \mu$ on

$\pi(\mathbb{R}) = \{(-\infty, b] : b \in \mathbb{R}\}$, which is a π-system

$\Rightarrow$ $\mathcal{P}_X = \mu$ on $(\mathbb{R}, \sigma(\pi(\mathbb{R}))) = (\mathbb{R}, \mathcal{B})$

$\Rightarrow$ $\mathbb{E}\, h(X) = \int_{-\infty}^{+\infty} h(x) f_X(x)\, dx$ for all indicator functions $h = I_B : \mathbb{R} \to \mathbb{R}$

$\Rightarrow$ True for all simple functions

$\Rightarrow$ True for all $h \geq 0$ $\mathcal{B}$-measurable

(By non. conv. THM. for $(\mathbb{R}, \mathcal{B}, \mathcal{P}_X)$ and $(\mathbb{R}, \mathcal{B}, \mathcal{L}|_{\mathcal{B}})$)

$\Rightarrow$ True for any $|h| \geq 0$. Thus, h $\mathcal{P}_X$ integrable $\Leftrightarrow$ hf $\mathcal{L}$-integr.

$\Rightarrow$ True for all $h : \mathbb{R} \to \mathbb{R}$ $\mathcal{B}$-measurable ■

(True for all $\mathbb{C}$-valued functions)

(Can use same if $X : (\Omega, \mathcal{F}, \mathbb{R}) \longrightarrow (S, \mathcal{B}_S)$, where $S \subseteq \mathbb{R}$. In this case $f_X(x) \equiv 0$ on S^c.)