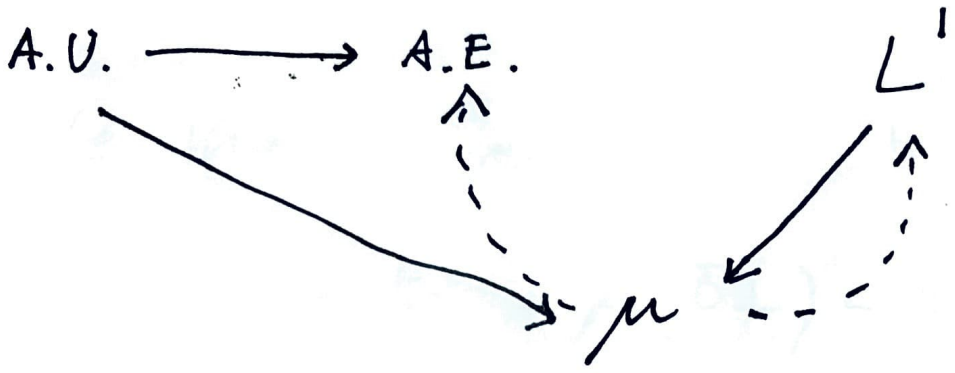
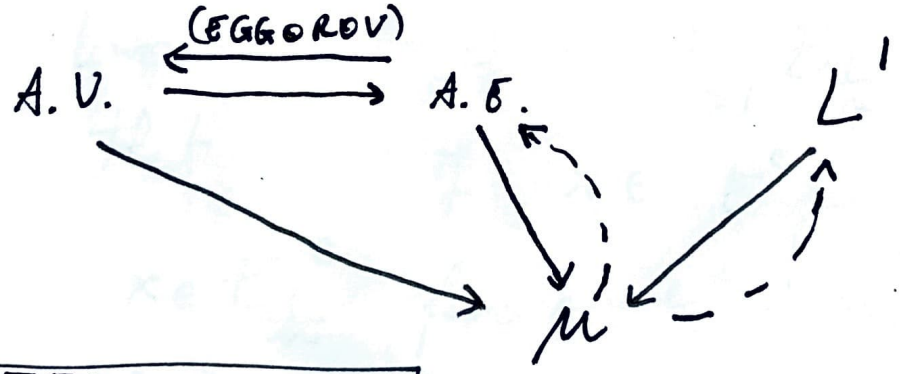
• GENERAL SPACE $(X, \mathcal{F}, \mu)$:



• FINITE SPACE $(X, \mathcal{F}, \mu)$, $\mu(X) < \infty$:



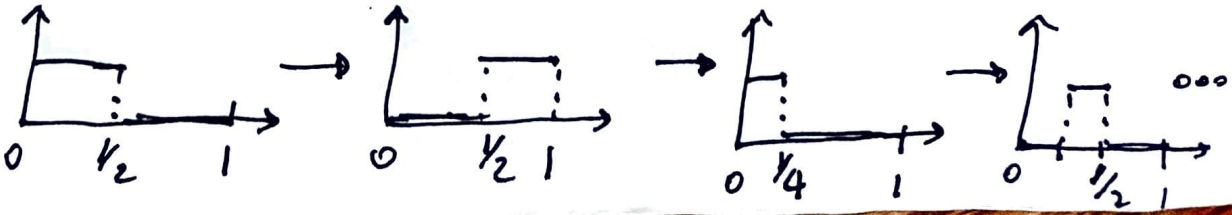
• COUNTEREXAMPLES:

(i): $\frac{1}{n}\,\chi_{(0,n)}$

(ii): $n\,\chi_{[0,\frac{1}{n})}$

(iii): $\chi_{(n,n+1)}$

(iv):



GENERAL SPACE :

A.U. $\Rightarrow$ A.E. :

We know A.U. $\Rightarrow$ $\forall n \geq 1$, $\exists E_{\frac{1}{n}}$ s.t. $\mu(E_{\frac{1}{n}}) < \frac{1}{n}$ and

$$f_n \longrightarrow f$$

UNIFORMLY on $E_{\frac{1}{n}}^c$.

$\rightsquigarrow$ Consider $E = \bigcap_{n \geq 1} E_{\frac{1}{n}}$. Note that if $x \in E^c = \bigcup_{n \geq 1} E_{\frac{1}{n}}^c$, then $x \in E_{\frac{1}{n}}^c$ for some $n \geq 1$.

~~$f_n(x) \to f(x)$ UNIFORMLY (HENCE ...)~~

$\rightsquigarrow$ $f_n(x) \to f(x)$. Now show $\mu(E) = 0$, so that $f_n(x) \to f(x)$ A.E. :

$$0 \leq \mu(E) < \mu(E_{\frac{1}{n}}) < \frac{1}{n}, \quad \forall n \geq 1$$

$$\Rightarrow \mu(E) \equiv 0 \quad \blacksquare$$

• A.U. $\Rightarrow$ μ :

Suppose not μ. Then, $\exists$ subseq. f_{n_k} s.t.

$$\mu\{|f_{n_k}(x) - f(x)| \geq \varepsilon\} \geq \delta > 0$$

for all $k \geq 1$ and some $\delta > 0$.

We know A.U. $\Rightarrow$ $\exists$ E_δ s.t. $\mu(E_\delta) < \delta$ and $f_n \to f$ UNIF. on E_δ^c.

Clearly, $|f_{n_k}(x) - f(x)| \geq \varepsilon$ $\forall$ $k \geq 1$

$\Rightarrow$ $f_{n_k}(x) \not\to f(x)$ $\Rightarrow$ $x \notin E_\delta^c$

$\Rightarrow$ $x \in E_\delta$ $\Rightarrow$ $\{|f_{n_k}(x) - f(x)| \geq \varepsilon\} \subset E_\delta$

$\Rightarrow$ $0 < \delta \leq \mu\{|f_{n_k}(x) - f(x)| \geq \varepsilon\}$

$\leq \mu(E_\delta) < \delta$. $\Rightarrow\Leftarrow$.

Thus, $f_n \xrightarrow{a.e.} f$ ■

• FINITE SPACES, $\mu(X) < \infty$:

• A.S. $\Rightarrow$ μ :

W.l.g., assume $f_n(x) \longrightarrow f(x)$ everywhere

$$\rightsquigarrow \bigcap_{N\geq 1} \bigcup_{n\geq N} \{|f_n(x) - f(x)| \geq \varepsilon\} = \emptyset,$$

and $\{|f_N(x) - f(x)| \geq \varepsilon\} \subset \bigcup_{n\geq N} \{|f_n(x) - f(x)| \geq \varepsilon\}$

Since $\mu(X) < \infty$, then can use continuity from above:

$$0 = \mu\left\{ \bigcap_{N\geq 1} \bigcup_{n\geq N} \{|f_n(x) - f(x)| \geq \varepsilon\} \right\} = \lim_{N\to\infty} \mu\left(\bigcup_{n\geq N} \{|f_n(x) - f(x)| \geq \varepsilon\} \right)$$

$$\geq \lim_{N\to\infty} \mu\{|f_N(x) - f(x)| \geq \varepsilon\}$$

$$\geq 0$$

$\rightsquigarrow$ μ-convergence ■