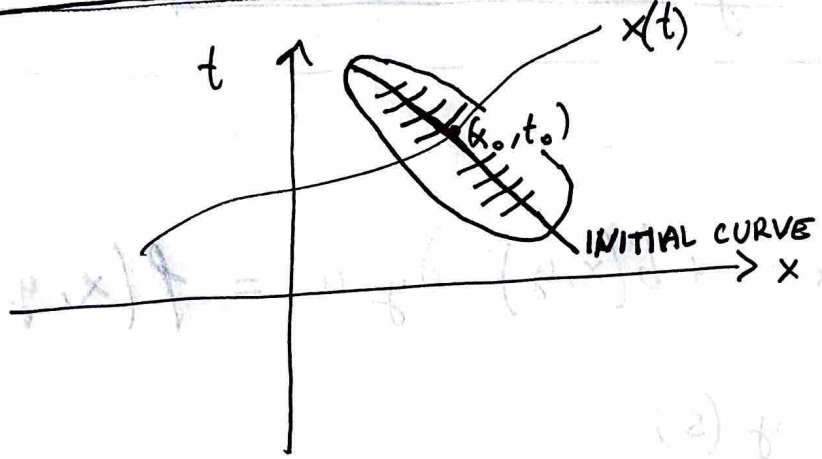
- **PDE**: (1st Class)

- $$a(x,t)\,\partial_t u + b(x,t)\,\frac{\partial u}{\partial x} = f(x,t)$$

$$\Rightarrow \quad \partial_t u + b(x,t)\,\partial_x u = f(x,t) \qquad \left(\text{Divided by } a(x,t)\right)$$

↝ METHOD OF CHARACTERISTICS:



↝ $$\underbrace{\frac{d}{dt}\,u\big(x(t),t\big)}_{\text{DIRECT. DERIVATIVE}} = \frac{\partial u}{\partial t} + \dot{x}\,\frac{\partial u}{\partial x}$$

↝ $\dot{x} = b(x,t)$, $\quad D_t u = f(x,t)$

- Method of Characteristics fails if initial information is given ALONG one of the Characteristics.

- Previous Example, solution to $\frac{dx}{dt} = b(x,t)$ is our CHARACTERISTIC.

→ Want to generalize notion of CHARACTERISTIC.

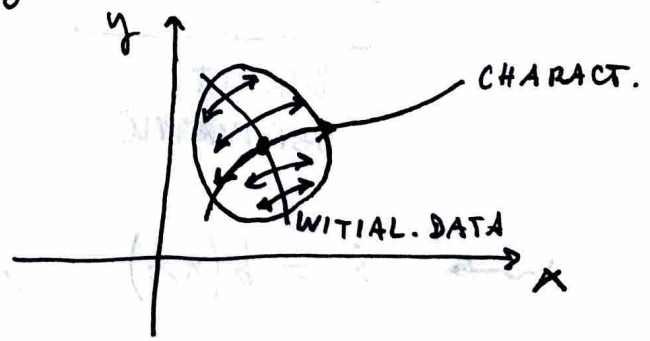
- Consider:

$$a(x,y)\,\partial_x u + b(x,y)\,\partial_y u = f(x,y,u).$$

Parametrize: $x(s)$, $y(s)$

$$\rightsquigarrow \quad D_s u = \dot{x}\,\partial_x u + \dot{y}\,\partial_y u$$

$$\rightsquigarrow \quad \begin{cases} \dot{x} = a(x,y) \\ \dot{y} = b(x,y) \end{cases}$$



$$\rightsquigarrow \quad D_s u = f(x,y,u)$$

- **MULTI-INDEX NOTATION:**

$$x = (x^1, x^2, \dots, x^m)$$

$$\frac{\partial u}{\partial x^i} = \partial_{x^i} u = u_{x_i}$$

$$a^i \partial_i u = \sum_{i=1}^{m} a^i \partial_i u \quad , \text{ i.e., } \underline{\text{REPEATED INDICES ARE SUMMABLE.}}$$

$$x^\alpha = |x^1|^{\alpha_1} |x^2|^{\alpha_2} \dots |x^m|^{\alpha_m}$$

- $a^i \partial_i u = f(x,u)$, $a(x,u)$ a vector ,

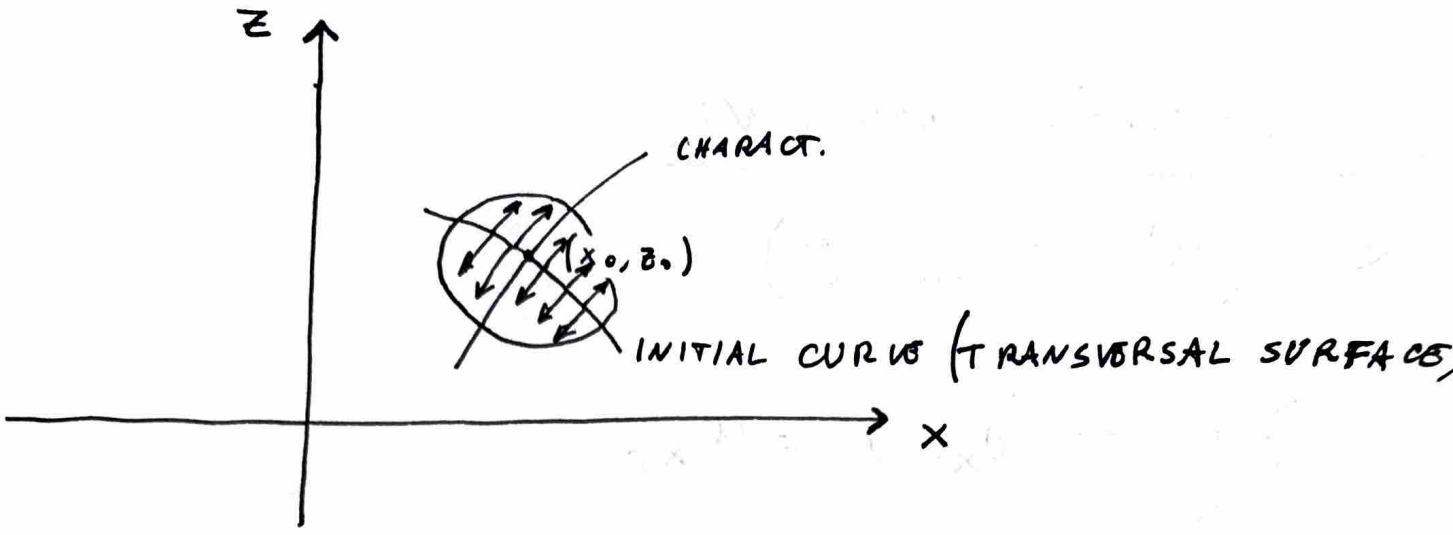
NON-LINEAR EQUATION, 1st ORDER.

- CHARACTERISTIC CURVE $x = \gamma(t)$.

$$\rightsquigarrow \quad D_t u = \dot{x} \nabla u = a^i \partial_i u = f$$

$$\rightsquigarrow \quad \underline{m+1:} \quad \begin{cases} \dot{x} = a(x,z) \quad , \text{ where } z = u(\gamma) \\ \qquad\qquad\qquad\quad \dot{z} = \dot{\gamma} \cdot \nabla u \\ \dot{z} = f(x,z) \quad , \text{ and } x(0) = x_0 \\ \qquad\qquad\qquad\quad z(0) = z_0 \end{cases}$$

INITIAL VALUE PROBLEM for ODE.



- If a & f are LIPSCHITZ, ∃! solution to system of ODE (locally)!

↝ Given transversal initial data, can construct a local solution of my PDE.

- Consider 2nd ORDER EQN:

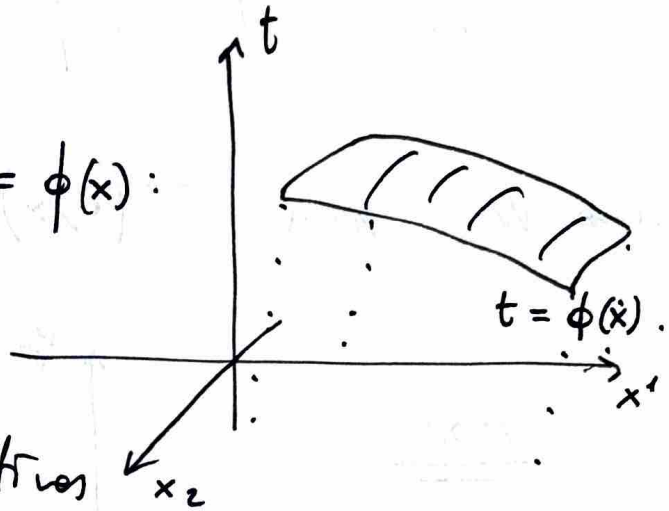
$$a(x,y)\,\partial_x^2 u + 2\,b(x,y)\,\partial_x\partial_y u + c(x,y)\,\partial_y^2 u = f(x,y).$$

(or $a^{ij}\partial^{ij}u = f$) . If I DON'T WORRY ABOUT UNIQUENESS, f CONTINUOUS ⇒ ∃ LOCAL SOLUTION to 1st ORDER PDE

- If f is continuous, $u \in C^1$ so 1st ORDER PDE's can be SOLVED. BUT $u \notin C^2$, hence problem with 2nd ORDER PDE.

2nd ORDER PDE: $a^{ij} \partial_i \partial_j u = f$

u given on surface $t = \phi(x)$:



→ Can compute all derivatives on surface BUT NOT the TRANSVERSAL ones!!

- $a(x,y)\partial_x^2 u + 2b\,\partial_x \partial_y u + c(x,y)\,u = f(x,y)$

  $t = \phi(x,y)$
  $s = x$
  } Change of Variables.

  ⇝ $u(x,y) = \bar{u}(\phi(x,y), s) = \bar{u}(\phi(x,y), x)$.

  ⇝ We are given $\bar{u}(0,s) = \bar{u}(0,x)$,
  and can compute all tangential derivatives of u on $\phi(x,y) = 0$,
  and also given $\nabla \bar{u}(0,x)$.

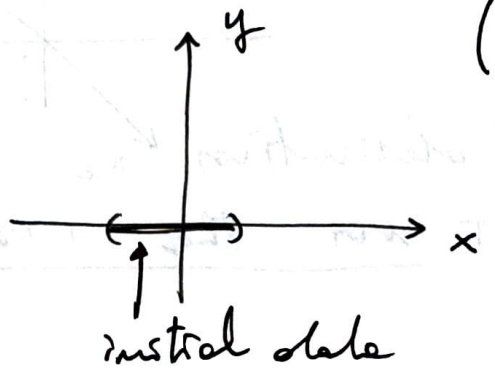
...

$$\rightsquigarrow \quad \frac{\partial u}{\partial x} = \phi_x \frac{\partial \bar{u}}{\partial \phi} + \frac{\partial \bar{u}}{\partial x}$$

$\rightsquigarrow$ We know $\bar{u}(0,x)$ and $\nabla \bar{u}(0,x)$.

$$\nabla \bar{u}(0,x) = (u_x, u_y, \dots).$$

<u>EX:</u>



$\phi(x,y) = y$, initial curve at $y = 0$.

Data Given at $y=0$:

- $u(x,0) \Longrightarrow \frac{\partial u}{\partial x}(x,0) \Rightarrow u_{xx}$ ✓
- $\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial \phi}(x,0) \cdot \frac{\partial \phi}{\partial y} \Rightarrow u_{xy}$ ✓ (derivatives commute).

- $u_{yy} = ?$ let's use the PDE to find it:

$$a(x,0)\underbrace{\partial_x^2 u(x,0)}_{\checkmark} + b(x,0)\underbrace{\partial_x \partial_y u(x,0)}_{\checkmark} + c(x,0)\partial_y^2 u(x,0) = f(x,0)$$

$\Rightarrow$ If $c \neq 0$, we can find $\partial_y^2 u(x,0)$!!

<u>and ALL HIGHER DERIVATIVES !!</u>

$\Rightarrow$ $\begin{cases} c \neq 0 & \text{NON-CHARACTERISTIC !!} \\ c = 0 & \text{IS A CHARACTERISTIC !!} \end{cases}$ $\Rightarrow$ CAN FIND POWER SERIES EXPANSION FOR $u|_{\phi=0}$ around $\phi = t = 0$.

↳ Want to do the same in many variables: $(x \in \mathbb{R}^n)$

$$\begin{cases} a^{ij}\partial_i \partial_j u = f, & \phi(x) = 0 \text{ initial surface.} \\ u|_{\phi=0} = u_0 \\ \dfrac{\partial u}{\partial \phi}\Big|_{\phi=0} = u_1 \quad \text{(given).} \end{cases}$$

$$\rightsquigarrow \quad \frac{\partial u}{\partial x^i} = \phi_{x^i} \frac{\partial \bar{u}}{\partial \phi} + \frac{\partial s^k}{\partial x^i} \frac{\partial \bar{u}}{\partial s^k} \quad \text{- where}$$

we changed $x \longmapsto (\phi, s^1, \dots, s^{n-1})$.

$$\frac{\partial^2 u}{\partial x^i \partial x^j} = \phi_{x^i} \phi_{x^j} \underbrace{\frac{\partial^2 \bar{u}}{\partial \phi^2}} + \text{Known}.$$

Only Unknown! $\Rightarrow$ USE PDE:

$$\Rightarrow \boxed{a^{ij} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j}} \frac{\partial^2 \bar{u}}{\partial \phi^2} + \text{Known} = f$$

(*)

... $\Rightarrow$ When $\circledast \neq 0$, can compute 2nd DERIVATIVES and all higher order ones.

$\Rightarrow$ $\boxed{a^{ij}\partial_i\phi\,\partial_j\phi = 0}$ is the CHARACTERISTIC CURVE

If $a^{ij}\partial_i\phi\,\partial_j\phi \neq 0$, can find Taylor Series expansion for u. If want this Taylor Series to converge, NEED u_0, u_1, a^{ij} & f to be ANALYTIC and also $\phi = 0$ to be an analytic NATURAL curve.

If we are given data on $\phi = 0$, ϕ analytic,

$$u\big|_{\phi=0} = u_0, \qquad \nabla\phi \cdot \nabla u\big|_{\phi=0} = u_1 \text{ given}$$

and if a^{ij} and f are ANALYTIC and if $\phi = 0$ is NON-CHARACTERISTIC, then we can find a formal POWER SERIES SOLUTION

REMINDER: $\partial^\kappa(fg) = (\partial^\kappa f)g + f(\partial^\kappa g) + \text{l.o.t.'s}$

- If I know derivatives of $(\kappa-1)$-th order and $\partial^\kappa g$, only unknown is $\partial^\kappa f$!!

THEOREM: (Cauchy - Kovalevski)

For this problem, the power series converges if $a^{ij}\phi = 0$, u_0, u_1 are ANALYTIC.

- **C-K**:

Given an analytic PDE with analytic data on an analytic surface that is non-characteristic, then $\exists$ LOCAL, UNIQUE ANALYTIC SOLUTION.

(UNIQUENESS comes from fact that INITIAL CONDITIONS + PDE UNIQUELY DETERMINES ALL TRANSVERSAL DERIVATIVES, since all functions are analytic!!!!)

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \quad A \cdot Du + Bu = F$$

$\det(A \cdot D\phi) \neq 0$ is a NON-CHARACTERISTIC

where $A \cdot D\phi = \left(A^{ijk}\partial_i \phi_k\right)$.

$$a^{i_1 \cdots i_n}\partial_{i_1} \cdots \partial_{i_n} u + \text{l.o.t.} = 0$$

$$\Rightarrow \left(a^{i_1 \cdots i_k}\phi_{x^{i_1}} \cdots \phi_{x^{i_k}} = 0\right).$$