PDE : (Class #2)

• ODE KNOWLEDGE:

$$\dot{x} = F(x), \qquad x \in \mathbb{R}^n.$$
$$x(0) = x_0$$

(i): If F is CONTINUOUS, $\exists$ solution in $[0,T]$ (locally)

(ii): If F is LIPSCHITZ, $\exists!$ " " "

(iii): $\sum_{k=0}^{\ell} a_k x^{k}(t) = f$, $\quad x^k = \partial_t^k x$

↳ Can be written as a system:

$$y_1 = x$$
$$y_2 = x^{(1)}$$
$$\vdots$$
$$\dot{y}^{\ell} = x^{(\ell-1)}$$

$$\Rightarrow \boxed{\dot{y} = F(y)}$$, need ℓ pieces of data, $x(0), \dots, x^{\ell-1}(0)$.

- CAUCHY PROBLEM:

Given a PDE of order k with data on the function and $(k-1)$ derivatives, FIND A LOCAL SOLUTION.

EX: (i): $u_{xx} = f(x)$ (ODE)

(ii): $u_{xx} + u_{yy} = f(x)$.

$u\big| = u_0$, $Du\big| = u_1$

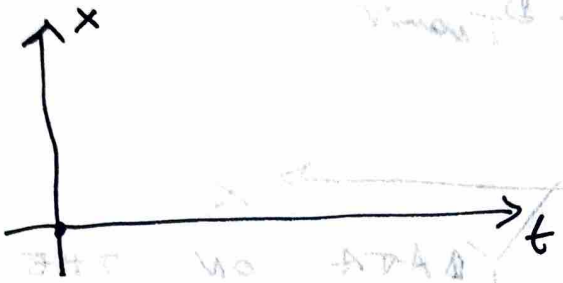
(iii): $u_{xy} = f(x,y)$

$u\big| = u_0$, $Du\big| = u_1$.

→ For Cauchy problems in PDE, Eqn's of order k, need k pieces of data

· **Ex:** $x'' + a x' + b x = f(t)$ (ODE)

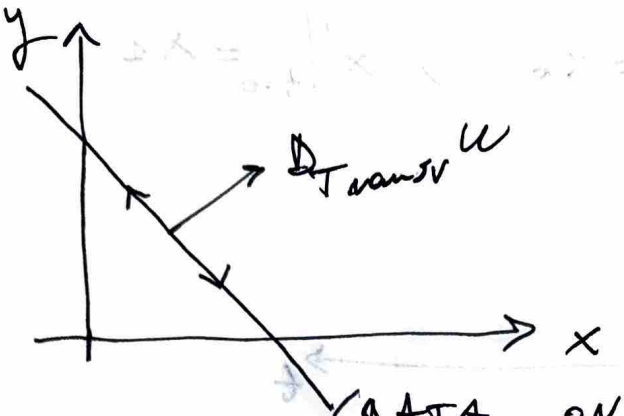
$x|_{t=0} = x_0$, $x'|_{t=0} = x_1$



→ Can use ODE to find x'' with initial data.
Can differentiate ODE again to find x''' and so on...
and can construct POWER SERIES SOLUTION ⟹ CAUCHY PROBLEM!!

- **Characteristic:**

- Ex: $u_{xy} = f(x,y)$



1. DATA ON THE CURVE
   ⇒ KNOW ALL TANGENTIAL DERIVATIVES of u.
   (u and $D_{tan} u$).

2. ⇒ Need to be given a TRANSVERSAL DERIVATIVE to get off the INITIAL CURVE. ($D_{Transv.} u$

⇒ Need a COORDINATE SYSTEM s.t. PDE equation is given on $0 = t = \phi(x,y)$. One variable is now t. Second variable $z = g(x,y)$.

⇒ Data now given as

$$u\Big|_{t=0} = u_0(z) \quad , \quad \partial_t u\Big|_{t=0} = u_1(z)$$

→ ooo

Know therefore

$$\partial_z^{(k)} u\big|_{\phi=0} \quad , \quad \partial_z^{\ell}\, \partial_t u\big|_{\phi=0} \quad , \quad u\big|_{\phi=0}$$

$\Rightarrow$ • Know all 1st order derivatives of u on $\phi=0$
• Know derivatives w.r.t. z on $\phi=0$
• Know all derivatives $\partial_z^{\kappa}\, \partial_t u\big|_{\phi=0}$

$\Rightarrow$ NEED u_{tt} !! USE PDE !!

$\rightsquigarrow$ $u(x,y) = \tilde{u}(t,z)$ (May drop $\sim$ for convenience)

$\rightsquigarrow$
$$\begin{cases} \dfrac{\partial u}{\partial x} = \phi_x \dfrac{\partial \tilde{u}}{\partial t} + \dfrac{\partial z}{\partial x}\dfrac{\partial \tilde{u}}{\partial z} \\ \dfrac{\partial u}{\partial y} = \phi_y \dfrac{\partial \tilde{u}}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial \tilde{u}}{\partial z} \end{cases}$$

↳ If take second derivatives, know everything but NOT $\dfrac{\partial^2 \tilde{u}}{\partial^2 t}$

$\Rightarrow$ $u_{xy} = \phi_x \phi_y \tilde{u}_{tt} + \underline{\text{KNOWN}}$!!

$\Rightarrow$ Plug it back into PDE.

... $\phi_x \phi_y \underbrace{\tilde{u}_{tt}}_{\downarrow} + \text{known} = \text{known}.$

Can solve for $\tilde{u}_{tt}$ if

$$\boxed{\phi_x \phi_y \neq 0}$$

ONLY CAVEAT IN CONSTRUCTING POWER SERIES SOLUTION !!

Same procedure to find 3rd derivatives



$L(x, \partial_x) u = f(x, y)$, linear, scalar equation.

$L(x, \xi) = \sum_{\substack{I=(i_1,\dots,i_m) \\ |I| \leq K}} a_I \xi^I$, linear, K^{TH} order and m variables (m = # of indices) (MULTI-INDEX NOTATION)

$k = 2$:

$$L(x, \partial) u = \sum_{0 \le \alpha+\beta \le 2} a^{ij}_{\alpha\beta} \, \partial_i^\alpha \partial_j^\beta u$$

$$= \sum_{i,j=1}^{n} a^{ij} \partial_i \partial_j u + \sum_{k=1}^{n} b_k \partial_k u + c u$$

$$\Rightarrow \begin{cases} t = \phi(x) \\ y^k = g^k(x) \end{cases}$$

$$\Rightarrow \partial_i u = \phi_i \frac{\partial u}{\partial t} + g^k_i \frac{\partial u}{\partial y^k}$$

$$\dots \quad \boxed{\sum_{i,j} a^{ij} \phi_i \phi_j = 0}$$

CHARACTERISTIC EQN, only related to highest order derivatives !!

- For general k:

$$L_k = \text{HIGHER ORDER TERM} =$$

$$= \sum_{|I|=k} a_I(x)\, \partial^I$$

$$\Rightarrow \quad \sum a_I(x)\, \phi_I =$$

$$= \sum_{|I|=k} a_I(x)\, \phi_I$$

$$= \sum_{1 \le |(i_1,\dots,i_k)| \le m} a_{i_1,\dots,i_k}(x)\, \phi_{i_1} \phi_{i_2} \cdots \phi_{i_k}$$

EX:

$$a(x,y)\, \partial_x^2 u + 2b(x,y)\, \partial_x \partial_y u + c(x,y)\, \partial_y^2 u + d u_x + e u_y + f = g$$

$$\Rightarrow \boxed{a(x,y)\, \phi_x^2 + 2b \phi_x \phi_y + c\, \phi_y^2 = 0}$$

CHARACTERISTIC EQUATION.

$\Rightarrow$ Assume $a \neq 0$. $\Rightarrow \dots$

... Can solve as:

$$\phi_x = \frac{-b \pm \sqrt{b^2 - ac}}{a}\,\phi_y.$$

Checking that surface is non-CHARACTERISTIC, just plug surface eqn into characteristic eqn and check $\neq 0$!!

(i): If $b^2 - ac < 0$, EQN. is ELLIPTIC. Can always find power series solution.

(ii): If $b^2 - ac > 0$, EQN. is HYPERBOLIC. 2 characteristic

(iii): If $b^2 - ac = 0$, EQN. is PARABOLIC. 1 characteristic.

- In N-dimension ...
  - If characteristic eqn has N SOLUTIONS $\Rightarrow$ HYPERBOLIC
  - If 0 SOLUTIONS $\Rightarrow$ ELLIPTIC
  - In between $\Rightarrow$ "no name".

- Without analyticity there cannot be general existence theorem for solutions in general (C^∞ NOT ENOUGH)

C-K: Requires coefficients, surface, and data to be ANALYTIC.

- **LAPLACE'S EQN.**:

$$\Delta u = \sum_{i=1}^{n} \partial_i^2 u = f$$

- **HARMONIC FUNCTIONS**: $\Delta u = 0$, $u \in C^2(\Omega)$, Ω open.

- **INTEGRATION BY PARTS**:

$$\int_\Omega a^i \partial_i f = -\int_\Omega (\partial_i a^i) f + \int_{\partial\Omega} (n \cdot a) f$$

- How do I check if a fn f is L^2?
  → Use duality. If it integrates against any other L^2 fun, then $f \in L^2$.

$$\int_\Omega \Delta u v = \int_\Omega \left(\sum_i \partial_i^2 u\right) v =$$

(By parts) $= -\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega}$

(By parts on 1st term) $= \int_\Omega u \Delta v - \int_{\partial\Omega} u (\nabla$

GREEN'S IDENTITY (Need $\partial\Omega$ orientable & Ω ...

$$\Rightarrow \int_\Omega (\Delta u) v - u(\Delta v) = \int_{\partial\Omega} (n \cdot \nabla u) v - (n \cdot \nabla v) \Omega$$

Want to know if u is HARMONIC, v is C^∞ (not a problem since C^∞ is dense in any L^p, $1 \le p < \infty$).

If u is HARMONIC, $\Delta u = 0$, if v is also H. then $\Delta v = 0$. If both are H. then for any A where H. holds

$$\int_{\partial A} (n \cdot \nabla u) v = \int_{\partial A} (n \cdot \nabla v) u$$

- **THM**: If u is HARMONIC on Ω, $\partial A \subset \Omega$, then $\int_{\partial A} n \cdot \nabla u = 0$

PROOF: Calculation above with $V = 1$. ∎

- **POLAR COORDINATES**:

2-D: (r, θ)

$$\Delta u = \frac{\partial^2}{\partial r} u + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \partial_\theta^2 u$$

3-D: (r, θ, φ), ω

$$\Delta u = \frac{\partial^2 u}{\partial r} + \frac{2}{r} \frac{\partial u}{\partial r} +$$

$$+ \frac{1}{r^2} \left(\partial_\varphi^2 u + \cot(\varphi) \frac{\partial u}{\partial \varphi} + \frac{1}{\sin^2(\varphi)} \frac{\partial^2 u}{\partial \theta} \right)$$

$\vdots$

n-D: $r = |x|$.

$$\omega = \frac{x}{|x|}.$$

$$\Rightarrow \Delta u = \frac{\partial^2}{\partial r} u + \frac{n-1}{r} + \frac{1}{r^2} \underbrace{(\Delta_\omega u)}_{\text{Angular Derivatives only}}$$

Solutions to $\Delta u = 0$?

- **2-D:**

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0.$$

Look for RADIAL SOLUTIONS ($u_{\theta\theta} = 0$)

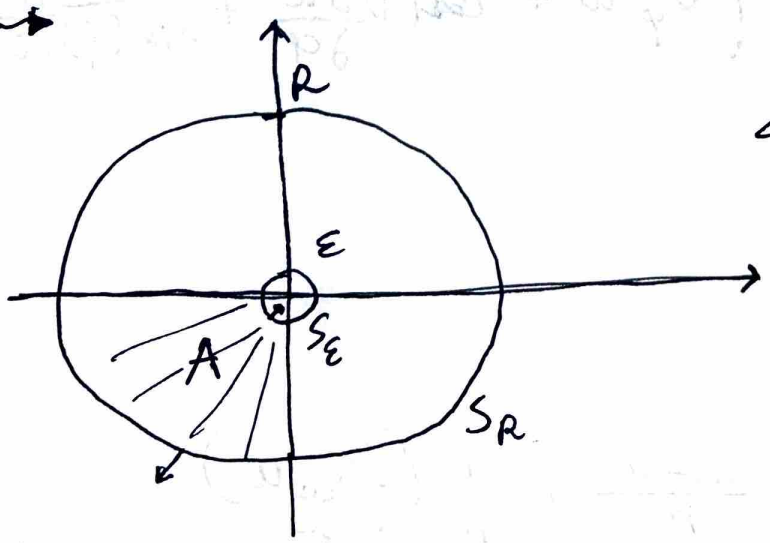
$$\Rightarrow u_{rr} + \frac{1}{r} u_r = 0$$

$$\Rightarrow \boxed{u = \ln(r)} \rightarrow \text{Bad solution across origin!}$$

- **3-D:** $u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \Delta_\omega u = 0$

Look for sol. s.t. $\Delta_\omega u = 0$:

$$\Rightarrow \boxed{u = \frac{c}{r}} \rightarrow \text{Bad solution across origin!}$$



$\Delta u = 0$ everywhere

$\Delta \underbrace{\frac{1}{r}}_{\| \; \vee} = 0$, $r \neq 0$,

$r = |x|$

$\omega = \frac{x}{|x|}$

...

$$\partial_i\left(\frac{1}{r}\right) = \partial_i\left(\frac{1}{\left(\sum_i x_i^2\right)^{1/2}}\right) = -\frac{x_i}{|x|^3}$$

... $V = \frac{1}{r}$

$$n \cdot \nabla V\Big|_{S_R^2} = \left(\frac{1}{r}\right)^2\Big|_{r=R} = -\frac{1}{R^2}$$

$$n \cdot \nabla V\Big|_{S_\varepsilon^2} = +\frac{1}{\varepsilon^2}$$

$$\Rightarrow \int_{\partial A} (n \cdot \nabla V)\, u = 0 \qquad \text{(Green's)}$$

$$0 = -\frac{1}{R^2}\int_{|\omega|=1} u(R\omega)\, R^2\, d\omega + \frac{1}{\varepsilon^2}\int_{|\omega|=1} u(\varepsilon\omega)\, \varepsilon^2\, d\omega$$

Take limit & since $u \in C$ on $|\omega|=1$ compact can bring limit inside

$$\Rightarrow \boxed{\int_{|\omega|=1} u(R\omega)\, d\omega = 4\pi\, u(0)}$$

MEAN VALUE THM. FOR HARMONIC. FUNCTIONS

- **M.V. THM** : u is HARMONIC on Ω ($u \in C^2(\Omega)$, $\Delta u = 0$ on Ω),
Then, $\forall\, x \in \Omega$, $B_R(x) \subset \Omega$

$$u(x) = \fint_{|w|=1} u(x+Rw)\, dw$$

$$= \frac{1}{\omega_n} \int_{|w|=1} u(x+Rw)\, dw,$$

(where $\omega_n = \frac{\text{Surface area of sphere in } n\text{-dim of radius } 1}{}$.)

written also

$$u(x) = \fint_{|y|=R} u(x+y)\, dS_R$$

$$= \frac{1}{\omega_n R^{n-1}} \int_{|y|=R} u(x+y)\, dS(y)$$

$\Rightarrow$ Integrate u over $R^{n-1}\, dR$

...

$$\dots \int_{B(z,R)} u(x)\,dx = \int_0^R \int_{|y|=r} u(z+y)\,dS(y)\,dr$$

(By M.V.P. above)

$$= \int_0^R |\partial B(z,r)|\,dr \quad u(z)$$

$$= |B(z,R)| \quad u(z) \quad \blacksquare$$

$$\Rightarrow \quad u(z) = \fint_{B(z,R)} u\,dy.$$

- Go from (M.V.T. $\oplus$ $u \in C$) to $u \in C^\infty$: (u HARMONIC)

$\rightsquigarrow$ $\varphi \in C^\infty$ Bump function, up to me to choose.

$\rightsquigarrow$ $\int u(y)\,\varphi(x-y)\,dy =$

$$= \int u(x-z)\,\varphi(z)\,dz$$

$$= \int u(x-rw)\,\varphi(r)\,r^{n-1}\,dr\,dw$$

$$= \int \varphi(r)\,r^{n-1} \int u(x-rw)\,dw\,dr$$

M.V.T. $$= \int \varphi(r)\,r^{n-1}\,u(x)\,dr$$

$$= u(x)\cdot C \quad , \text{ take } \varphi \text{ s.t. } C=1.$$

• POLAR COORDINATES :

(i) : • Area element in 2-d :

↳ $r\, dr\, d\theta$

• Volume element in n-d

↳ $r^{n-1}\, dr\, d\omega$

$d\omega$ angles ...

(ii) : • ω_n = Surface area of $B(0, 1)$, in $\mathbb{R}^n$

$\omega_2 = 2\pi$

$\omega_3 = 4\pi$

$r^{n-1}\omega_n$ = Surface area of $B(0, r)$

• α_n = Volume of $B(0, 1) = \frac{\omega_n}{n}$, in $\mathbb{R}^n$

$\alpha_2 = \pi$

$\alpha_3 = \frac{4\pi}{3}$