• **Class**: (#3)

(1): Mean value theorem $\Rightarrow$ Liouville Theorem
$\Rightarrow$ Max Principle
$\Rightarrow$ Uniqueness of Dirichlet Problem

(2): Fundamental Solution

(3): Theory of Distributions

(4): Dirichlet & Neumann Problems

(5): Potential Theory.

• **THM**: If $u \in C^0(\Omega)$ satisfies the M.V.P., then u is harmonic and $u \in C^\infty$

PROOF: φ_ε our mollifiers, $\varphi_\varepsilon \in C_c^\infty$, φ_ε RADIAL FUNCTION.

$$\rightsquigarrow \int u(y)\,\varphi_\varepsilon(x-y)\,dy = \int u(x-y)\,\varphi_\varepsilon(y)\,dy$$

$$\overset{\text{M.V.P.}}{=} u(x)\int \varphi_\varepsilon(y)\,dy$$

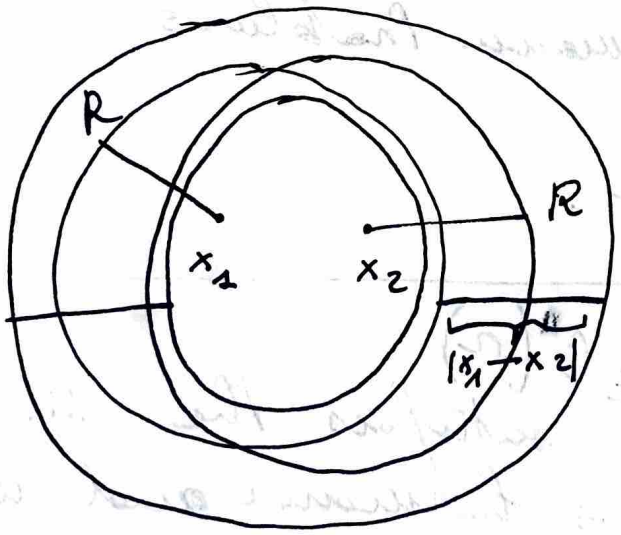
$\Rightarrow u(x) \in C^\infty$ (Verify $\Delta u = 0$ in same way)

- **LIOUVILLE THM**: on $\mathbb{R}^n$.

Bounded, Harmonic functions are just constants.

PROOF:

$$u(x_1) - u(x_2) = \fint_{B(x_1,R)} u - \fint_{B(x_2,R)} u \leq \dots$$



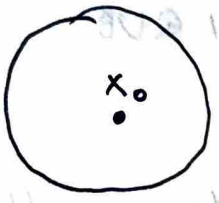
$$\dots \leq \fint_{R-|x_1-x_2|<|y|<R+|x_1-x_2|} |u| \leq C \fint_{R-|x_1-x_2|<|y|<R+|x_1-x_2|} 1$$

$$\leq \frac{C}{R^n}\left[(R+|x_1-x_2|)^n - (R-|x_1-x_2|)^n\right]$$

$$\approx \frac{R^{n-1}}{R^n}(\dots) \xrightarrow{R\uparrow\infty} 0 \quad \blacksquare$$

- **REMINDER**: Ω open, $u \in C^2(\Omega)$ and $\Delta u = 0$ on Ω. Then u is said to be Harmonic on Ω.

- Suppose u Harmonic, $u(x) = \fint_{B(x,r)} u$



If $u(x_0) = A = \sup_{x \in \Omega} u(x) = \fint_{B(x,r)} u$

$\Rightarrow$ $u = A$ on $B(x_0, r)$,

(otherwise $\fint_{B(x_0,r)} u < A$).

- **MAXIMUM PRINCIPLE**: Ω open & bounded.

(i): Ω open, u Harmonic on Ω, and $\max_{x \in \overline{\Omega}} u(x) = u(x_0) = A$, Then

$u = A \quad \forall\ x \in \Omega$.

(ii): ($\overline{\Omega}$ Compact) Ω open, u Harmonic on Ω and $u \in C(\overline{\Omega})$, then

$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$. (same for "min").

• **THM**: If $u \in C(\bar{\Omega})$ and u harmonic on Ω, then the problem

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u|_{\partial\Omega} = f \end{cases}$$

has a UNIQUE solution.

Proof: Suppose u_1, u_2 are solutions,

$$v = u_1 - u_2$$

$$\Rightarrow \begin{cases} \Delta v = 0 \\ v|_{\partial\Omega} = 0 \end{cases}$$

$$\Rightarrow \max(v) = \min(v) = 0$$

by the maximum principle

$$\Rightarrow v \equiv 0 .$$

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- $N(x) = \frac{1}{(2-n)\omega_n} \frac{1}{|x|^{n-2}}$, $\varphi \in C_c^2$

- $\int \Delta\varphi(x)\, N(x)\, dx = \varphi(0)$.

- $\int \varphi(x)\, "\Delta N"\, dx = \varphi(0)$.

$\Delta \frac{1}{|x|^{n-2}} \sim \frac{1}{r^n}$ ← Not integrable around zero.

- By Translation: $\int \Delta\varphi(y)\, N(x-y)\, dy = \varphi(x)$

⇝ By parts twice:

$$\int \varphi(y)\, "\Delta_x N(x-y)"\, dy =$$

$$= \Delta_x \int \varphi(y)\, N(x-y)\, dy$$

- If $\int \varphi(y)\, N(x-y)\, dy$ is C^2, then

$$\Delta \int \varphi(y) N(x-y)\, dy = \Delta \int \varphi(x-y)\, N(y)\, dy$$

$$= \int \Delta\varphi(x-y)\, N(y)\, dy = \varphi(x).$$

When is it true

$$\frac{\partial}{\partial x_i} \int \varphi(y)\, N(x-y)\, dy =$$

$$= \int \varphi(y) \frac{\partial}{\partial x_i} N(x-y)\, dy \quad ?$$

→ Need L^1 localed derivative !!

$$N = \frac{c}{|z|^{n-2}}$$

$$\partial_i \frac{1}{|z|^{n-2}} = (n-2) \frac{z_i}{|z|^n} \in L^1_{loc} \; ?$$

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$$\frac{1}{\rho^{n-1}} \rho^{n-2}\, d\rho$$

$$\partial_i^2 N \sim \frac{1}{\rho^n} \notin L^1_{loc}, \quad \text{it is not } L^1_{loc}$$

by a marginal amount, i.e., $\frac{1}{\rho^{n-\varepsilon}}$ is in L^1_{loc} for any small $\varepsilon > 0$.

⇒ Can try to check if some cancellation leads to L^1_{loc}.

↝ ...

- **1-dim:** $\varphi \in C^1$, $\int_{-1}^{1} \frac{1}{x} \varphi(x)\, dx$.

By Taylor's Thm.,

$$\varphi(x) = \varphi(0) + x R(x),$$

$$\lim_{x \to 0} R(x) = 0, \quad |R(x)| \leq C$$

$$\text{P.V.}\left(\frac{1}{x}\right) \tilde{=} \text{P.V.} \int_{-1}^{1} \frac{1}{x} \varphi(x)\, dx =$$

$$= \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{\varphi(x)}{x}\, dx + \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{\varphi(x)}{x}\, dx$$

$$= \lim_{\varepsilon \to 0} \left[\int_{\varepsilon}^{1} \frac{\varphi(0)}{x}\, dx + \int_{-1}^{-\varepsilon} \frac{\varphi(0)}{x}\, dx \right] +$$

$$+ \underbrace{\int_{-1}^{1} R(x)\, dx}_{\text{integrable}}$$

$$= \lim_{\varepsilon \to 0} \underbrace{\left[\varphi(0) \int_{\varepsilon}^{1} \frac{1}{x}\, dx - \varphi(0) \int_{\varepsilon}^{1} \frac{1}{x}\, dx \right]}_{=0}$$

$$+ \int_{-1}^{1} R(x)\, dx.$$

• In $\mathbb{R}^n$: $K(z)$, suppose

$$K(rz) = r^{-n} K(z)$$

(homogeneous of degree "$-n$")

$$\rightsquigarrow K(z) = K\left(|z|\,\omega\right) = |z|^{-n} K(\omega),$$

$$|\omega| = 1.$$

$$\Rightarrow K(z) = \frac{K(\omega)}{|z|^n}, \text{ NOT } L^1_{loc}.$$

$\rightsquigarrow$ Want to define (integration on BOUNDED DOMAIN Ω)

$$\text{P.V.} \int_\Omega f(z)\, K(z)\, dz =$$

$$= \lim_{\varepsilon \to 0} \int_{\Omega > |z| > \varepsilon} f(z)\, K(z)\, dz = \cdots$$

$$\ldots = \lim_{\varepsilon \to 0} \int_{b > r > \varepsilon} f(r\omega)\, \frac{K(\omega)}{r^n}\, r^{n-1}\, dr\, d\omega = \ldots$$

$$\left(f \text{ math} \Rightarrow f(0) + r\, R(r, \omega) \right)$$

$$\ldots = \lim_{\varepsilon \to 0} \int \left(\underbrace{f(0)\, \frac{K(\omega)}{r}}_{??} + \underbrace{R(r,\omega)\, K(\omega)}_{\text{INTEGRABLE } \checkmark} \right) dr\, d\omega$$

$$f(0) \log(\varepsilon) \underbrace{\int K(\omega)\, d\omega}_{:= \circledast}$$

$\rightsquigarrow$ The only way for this limit to make sense is to have $\circledast = 0$!!

- Thus, if

$$\int K(\omega)\, d\omega = 0 \,,$$

then

$$\text{P.V.} \int f(z)\, K(z)\, dz$$

is well-defined.

$\Longrightarrow \infty$

- $\int_{\varepsilon<|z|<b} C\cdot K(z)\,dz = \big(\log(b) - \log(\varepsilon)\big) \underbrace{\int_{|\omega|=1} K(\omega)\,d\omega}_{\| \; 0}$

$$\Rightarrow \text{P.V.}\int f(z)\,K(z) = \lim_{\substack{\varepsilon\to 0\\ b\to\infty}} \int_{\varepsilon<|z|<b} f(z)\,K(z)\,dz$$

$$= \lim_{b\to\infty} \int_{|z|<b} \big[f(z) - f(0)\big]\,K(z)\,dz,$$

i.e., can "replace" $\lim_{\varepsilon\to 0}$ by subtracting from $\text{P.V.}\int fK\,dz - f(0)\int K(z)\,dz$. For this to make sense, I repeat that

$$\boxed{\int_{|\omega|=1} K(\omega)\,d\omega = 0}$$

must hold!!

- $\text{P.V.}\int f(z)\,K(z)\,dz = \lim_{b\to\infty} \int_{|z|<b} \underbrace{\big[f(z)-f(0)\big]}_{\updownarrow} \underbrace{K(z)\,dz}_{\updownarrow}$

f smooth $\Rightarrow O(z)$ $\quad \approx \frac{1}{|z|^n}$

$\longrightarrow \cdots$

→ ... Thus, can generalize the above ideas to $[f(z) - f(0)] = O(z^\alpha)$, $\alpha \in (0,1)$

(i.e., $f \in C^\alpha \Rightarrow$ P.V. $\int f(z) K(z)\, dz$ STILL WELL-DEFINED !(!!)) → NO USED for f smooth

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$\alpha \in (0,1)$:

- $f \in C^\alpha(\bar{\Omega})$ if $\|f\|_{SUP} = |f|_C < \infty$, and

$$[f]_{C^\alpha} = \sup_{x,y \in \Omega} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty$$

- $C^\alpha(\bar{\Omega})$-SPACE: Can put NORM on it by

$$|f|_{C^\alpha} = |f|_C + [f]_{C^\alpha}$$

- $C^{k,\alpha}$ SPACE: Put NORM on it by

$$|f|_{C^{k,\alpha}} = |f|_{C^k} + |\partial^k f|_{C^\alpha}$$

where $C^k(\Omega)$ is space with norm: $|f|_{C^k} = \sup_{\substack{x \in \Omega \\ |\kappa| \le k}} |\partial^\alpha f(x)|$

- **NOTE:** $K : \mathbb{R}^n \longrightarrow \mathbb{R}$, then

$$K(rz) = \frac{K(z)}{r^n}, \quad \int_{0<a<|z|<b} K(z)\, dz = 0 \overset{(\forall\, a,b)}{\iff} 0 = \int_{|w|=1} K(w)\, dw$$

- **DEF:** $K : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a SINGULAR INTEGRAL OPERATOR if

(i): $K(rz) = \frac{K(z)}{r^n}$ and $\int_{a<|z|<b} K(z)\, dz = 0 \quad (\forall\, \overset{\circ}{a}, b)$

(ii): $K \in C^1(\mathbb{R}^n \setminus 0)$.

$$W(x) := Ku(x) := \lim_{\substack{\varepsilon \to 0 \\ b \to \infty}} \int_{\varepsilon<|y|<b} u(x-y)\, K(y)\, dy$$

$$= \lim_{b \to \infty} \int \left[u(x-y) - u(x) \right] K(y)\, dy$$

↳ In previous pages, we showed

$u \in C^\alpha \Longrightarrow w \in L^\infty$.

- **THM**: $u \in C^\alpha(\bar{\Omega}) \Rightarrow w \in C^\alpha$.

(use $w = g$).

PROOF: By above, $w(x)$ is well-defined.

- $h_1(x) := \int_{|z| < 3|y|} [g(x-z) - g(x)]\, u(z)\, dz$

- $h_2(x) := \lim_{b \to \infty} \int_{3|y| < z < b} [g(x-z) - g(x)]\, u(z)\, dz$

- $|h_1(x)| \le C \int_{|z| < 3|y|} |z|^{\alpha - n}\, dz = C \int_0^{3|y|} r^{\alpha - 1}\, dr$

$= C\, |y|^\alpha$

$\Rightarrow |h_1(x+y) - h_1(x)| \le |h_1(x+y)| + |h(x)| \le 2C|y|^\alpha$

- Further,

$$h_2(x+y) = \lim_{b \to \infty} \int_{|3y| < |z| < b} [g(x+y-z) - g(x)]\, u(z)\, dz$$

$$= \lim_{b \to \infty} \int_{3|y| < |z+y| < b} [g(x-z) - g(x)][u(z+y)]\, dz$$

$$\Rightarrow \quad h_2(x+y) - h_2(x) =$$

$$= \lim_{b\to\infty} \int_{3|y|<|z|<b} [g(x-z) - g(x)] \left[K(z+y) - K(z)\right] dz$$

$$+ \lim_{b\to\infty} E_1(b) + E_2 .$$

$$\cdot |E_1(b)| \le \int_{b-|y|<|z|<b+|y|} \|g\|_\infty \frac{1}{b^n} dz =$$

$$= C\|g\|_\infty \frac{(b+|y|)^n - (b-|y|)^n}{b^n} \xrightarrow{b\uparrow\infty} 0$$

since $E_1(b)$ is error coming from difference between region $|z|<b$ & $|z+y|<b$, which is bold by $b-|y|<|z|<b+|y|$.

Assuming $b >> y$, we used the fact that $|z| \simeq |z+y| \simeq b$

$$\Rightarrow \quad |K(z)|, |K(z+y)| \approx \left|\frac{K(\omega)}{b^n}\right| \le \frac{C}{b^n}$$

since $|\omega|=1$ is compact & $K \in C^2(\mathbb{R}\setminus 0)$.

E_2 comes from symmetric difference between regions $|z+y| > 3|y|$ & $|z| > 3|y|$, which is bold by $2|y| < |z| < 4|y|$. In this annulus we have $|z| \simeq |z+y| \simeq |y|$ so:

$$|E_2| \leq C \int_{2|y|<|z|<4|y|} |y|^\alpha \frac{1}{|y|^n} dz$$

$$= C |y|^\alpha$$

- Finally, the main term left above is bold by (use M.V.T. for K & C^α for u)

$$\cdots \leq C \int_{3|y|<|z|} |z|^\alpha \cdot \frac{|y|}{|z+\xi|^{n+1}} dz$$

$$\leq C |y|^\alpha \quad \blacksquare$$

THM: If $f \in C^\alpha$, then

$u = f * N$ is $C^{2+\alpha}$

and $\Delta u = f$.