• **Class #4**:

$u \in C^2(\Omega)$,
$\Delta u = 0$, $N(x-y) = \dfrac{c}{|x|^{n-2}}$

$$\rightsquigarrow \int_\Omega \Delta u\, N = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} N - \int_{\partial\Omega} u \frac{\partial N}{\partial \nu} + \int_\Omega u \Delta N$$

$$\left(\text{In the limit as } \varepsilon \to 0\right) \Rightarrow \quad 0 \qquad\qquad \Delta N \cong \delta(x) \Rightarrow -u(x)$$

$$\rightsquigarrow \quad u(x) = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} N - \int_{\partial\Omega} u \frac{\partial N}{\partial \nu} \qquad (*)$$

↳ Tempting because gives me solution in terms of $\frac{\partial u}{\partial \nu}$, u on $\partial\Omega$ = Cauchy Problem. But this is wrong, as we showed that $u|_{\partial\Omega}$ only already determines solution UNIQUELY !!

$\rightsquigarrow$ Consider $G(x,y) = -N(x-y) + K(x,y)$

$\rightsquigarrow$ $\Delta G = +\delta + \Delta K$. To insert G in $(*)$, need $\Delta_x K = 0$ on Ω. $\rightsquigarrow$

$$\Rightarrow u(x) = \int_{\partial\Omega} \frac{\partial G}{\partial \nu} u - \int_{\partial\Omega} \frac{\partial w}{\partial \nu} G$$

$\rightsquigarrow$ Suffice can find K s.t. $\Delta_x K = 0$

and $K|_{\partial\Omega} = N|_{\partial\Omega} \Rightarrow G|_{\partial\Omega} = 0$

$\rightsquigarrow$ $\boxed{u(x) = \int_{\partial\Omega} \frac{\partial G}{\partial \nu} u}$, (we dropped the terms containing $\frac{\partial u}{\partial \nu}$ "Degenerate")

- Such G is called the GREEN'S FUNCTION.

- As before, $\boxed{u = \int f(y) N(x-y)\, dy}$ ⊛

We showed $\rightsquigarrow$ $-\Delta u = -f$, and

if $f \in C^\alpha(\mathbb{R}^n) \Rightarrow u \in C^{2+\alpha}(\mathbb{R}^n)$.

- But ⊛ holds for $f \notin C^\alpha$, so may hope to generalize notion of solution.

- NB: $f \in C \not\Rightarrow u \in C^2$ (COUNTEREXAMPLE).

- **CAUCHY PROBLEM for Δ IS ILL-POSED:**

Ex: $u_{xx} + u_{yy} = 0$, $\quad u(x,0) = \delta \sin(nx)$, $\delta > 0$.

$\partial_y u(x,0) = 0$.

→ Give me data very close to zero, i.e. $\delta \cong \frac{1}{n}$, but as $|y| \to 1$ $|u| \to \infty$

(Can solve explicitly).

→ We showed in HW that (A.E.) $\Delta u = 0$ makes problem ill-posed.

- $f \in L^1$: 2 ways to say f is zero:

(i): $f = 0$ A.E.

(ii): $\int f \varphi \, dx = 0 \quad \forall \varphi \in C_0^\infty$

- $\Delta u = f$, suppose I have such $u \in C^2$.

$\rightsquigarrow \int \Delta u \, \varphi = \int f \varphi$

(By parts) $\Rightarrow$ $\boxed{\int u \Delta \varphi = \int f \varphi, \quad \forall \varphi \in C_0^\infty.}$ (△)

↳ Want to say "u" a solution if this holds.

→ No differentiability needed.

• NB: If f in (*) only locally integrable, then (Δ) holds.

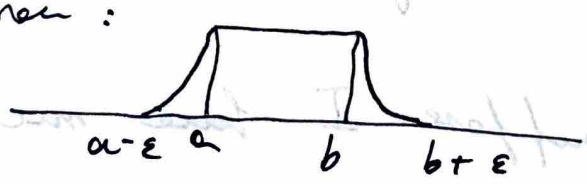
• $\varphi \in \underbrace{C_c^\infty(\Omega)}_{\| \; D(\Omega)}$, $\varphi \in C^\infty$, $\operatorname{supp}\varphi \subset\subset \Omega$, Ω open.

• If $\varphi \in C^\infty(a,b)$, by Taylor's Thm

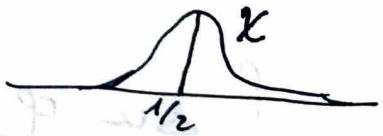
$$\varphi(x) = \sum_{i=0}^{k} \varphi^{(i)}(x_0)\frac{(x-x_0)^i}{i!} +$$

$$+ \frac{(x-x_0)^{k+1}}{k!}\int_0^1 (1-s)^k \varphi^{(k+1)}\big(x_0+s(x-x_0)\big)\,ds$$

• Want function:



Does it exist? Yes:



→ Take bump function $\chi \Rightarrow \chi_\varepsilon(x) = \frac{1}{\varepsilon}\chi\left(\frac{x}{\varepsilon}\right)$ and convolve it with identity function (Charact. function)

$$\chi_\varepsilon * \mathbb{1}_{[a-\frac{\varepsilon}{2},\, b+\frac{\varepsilon}{2}]}(x).$$

NOTATION: $A \subset\subset S$ = ("A compacts inside S.")

- **PARTITION OF UNITY:**

(i): $K \subseteq \bigcup_{i=1}^{N} O_i$, $\varphi_i \in C_0^\infty(O_i)$,

$$\sum_{i=1}^{N} \varphi_i(x) = 1, \text{ on } K.$$

(ii): $\bigcup_{i=1}^{\infty} O_i$, $\exists$ $\operatorname{supp}(\varphi_i) \subset\subset O_i$

$$\sum_{i=1}^{\infty} \varphi_i(x) = 1, \quad x \in \bigcup_{i=1}^{\infty} O_i.$$

- $C_0^\infty(a,b)$: Need notion of convergence on this space!

$u_m \xrightarrow{C_0^\infty} u$ if $\operatorname{supp}(u_m) \subset K \subset\subset (a,b)$

and $\forall k$ $\quad |u_m - u|_{C^k(K)} \longrightarrow 0$.

- $\ell \in \mathcal{D}'(a,b) = \left(C_0^\infty(a,b)\right)^*$:

(I): CONTINUITY: If $\varphi_m \xrightarrow{C_0^\infty} \varphi$, then

$\ell(\varphi_m) \longrightarrow \ell(\varphi)$

$\Updownarrow$

(II): BOUNDEDNESS: $\forall K \subset\subset (a,b)$, $\exists C_K$ and $N(K)$ s.t.

$\forall \varphi \in C_0^\infty$, $\operatorname{supp}(\varphi) \subset K \Rightarrow |\ell(\varphi)| \leq C_K |\varphi|_{C^{N(K)}}$

EX 1: $u \in L^1_{loc}$

$$\left| \int u \varphi \, dx \right| \leq \int_K |u| \, dx \; |\varphi|_{C(K)}$$

$$\Rightarrow \quad \ell_u(\varphi) = \int u \varphi \qquad \text{is a distribution}$$

NOTATION: $\langle u, \varphi \rangle := \ell_u(\varphi) = \int u \varphi$

EX 2: $\langle \ell, \varphi \rangle = \varphi(0)$,

we call such $\ell = \delta$ "Dirac Mass"

$$\text{or} \quad \langle \delta, \varphi \rangle := \varphi(0)$$

$$\text{or} \quad \langle \delta_{x_0}, \varphi \rangle := \varphi(x_0) .$$

$\rightsquigarrow$ δ is a distribution.

If T is a distribution, abuse of NOTATION and write

$$\langle T, \varphi \rangle \overset{\sim}{=} \int T \varphi$$

If T is a regular function, then $=$ is formally true. **EX:** $\int \delta \varphi = \varphi(0)$ is just NOTATION.

Ex 3: $\text{P.V.}\left(\frac{1}{x}\right)$ is also a distribution:

$$\left\langle \text{P.V.}\left(\frac{1}{x}\right), \varphi \right\rangle =$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}\setminus[-\varepsilon,+\varepsilon]} \frac{\varphi(x)}{x}\, dx$$

$$= \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{\varphi(x) - \varphi(-x)}{x}\, dx \quad \left(\begin{matrix}\text{By} \\ \text{Changing} \\ \text{variables}\end{matrix}\right)$$

We want to show it's a distribution:

$$\left| \left\langle \text{P.V.}\left(\frac{1}{x}\right), \varphi \right\rangle \right| \leq 2|c| \sup |\varphi'(x)| \quad , \text{ where}$$

$$c = \operatorname{supp}(\varphi).$$

By Taylor's Thm or Mean Value Thm.

- **Norm on $C^N(K)$:**

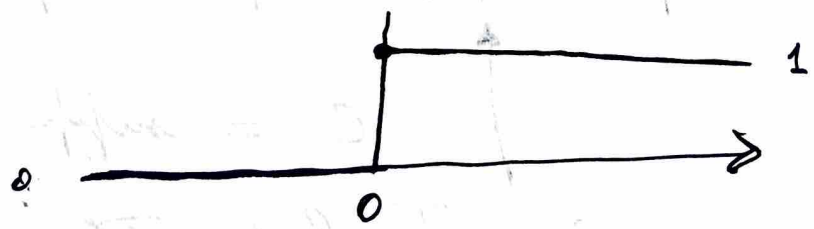
$$|u|_{C^N(K)} = \sup_{x \in K} \left\{ \sum_{m=0}^{N} |u^{(m)}(x)| \right\}$$

- $T \in \mathcal{D}'(a,b)$, want to define derivative of T:

$$\langle T', \varphi \rangle \overset{(def)}{:=} - \langle T, \varphi' \rangle \qquad \left(\text{notation } DT := T' \right)$$

- Any operation that can be done for $\varphi \in C_0^\infty$, can be extended to $T \in \mathcal{D}(a,b)$.

- **Ex 1:** "Heaviside Function" $H(x)$



$H'(x) = 0$ a.e., but

$DH = \delta$:

$$\langle DH, \varphi \rangle = - \int H \varphi'(x)\, dx = - \int_0^\infty \varphi'(x)\, dx = \varphi(0).$$

EX 2:

$$\frac{d}{dx} \ln|x| = \frac{1}{x}.$$

$$D \ln|x| = \text{P.V.}\left(\frac{1}{x}\right).$$

EX 3: $f \in L^1_{loc}$

$$\Rightarrow F(x) = \int_a^x f(t)\,dt$$ Continuous at least!!

$\Rightarrow F \in L^1_{loc} \Rightarrow$ F is a Distribution

$$\Rightarrow \langle DF, \varphi \rangle = -\langle F, \varphi' \rangle$$

$$= -\int F(x)\,\varphi'(x)\,dx$$

(Plug in F & use FUBINI)

$$= \langle f, \varphi \rangle$$

- In Real Variables $\exists$ function whose derivative is $= 0$ A.E. and that is increasing (Cantor non-constant Function)

$\Rightarrow$ Cantor Fn $\neq \int$ (Cantor Function)$'\,dx$

- With Distributions, on the other hand:

- **LEMMA**: If $T' = 0$, then

$$T = \text{CONSTANT}$$

$$(T \in \mathcal{D}'$$

PROOF:

- ψ, $\int \psi = 0$, ψ COMPACTLY SUPP. C^∞

$$\varphi(x) = \int_{-\infty}^{x} \psi(t)\,dt \qquad \in C_0^\infty.$$

- If $\int \psi \neq 0$, let $C_\psi = \int \psi$ and take $\chi \overset{\in C_0^\infty}{}$ s.t. $\int \chi = 1$.

$\Rightarrow$ Define

$$C_0^\infty \ni \varphi(x) = \int_{-\infty}^{x} (\psi - C_\psi \chi),$$

$$\rightsquigarrow \quad \langle T, \varphi' \rangle =$$

$$= \langle T, \psi - C_\psi \chi \rangle$$

$$\langle T, \psi \rangle = C_\psi \langle T, \chi \rangle$$

$$\int \psi\,dx \qquad = \text{constant}$$

■

. **LEMMA**: ($\approx$ FUND. THM. CALCULUS FOR $\mathcal{D}'$)

If $S \in \mathcal{D}'$, $\exists\ T \in \mathcal{D}'$ s.t. $DT = S$

PROOF: $\varphi \in C_0^\infty \longrightarrow M(\varphi)(x) = \int_{-\infty}^{x} \varphi(t) - C_\varphi \chi(t)\,dt$

(same map as previous LEMMA)

Also define $\langle T, \varphi \rangle \overset{(DEF)}{=} - \langle S, M(\varphi) \rangle$

→ Need to check $DT = S$:

$$\langle DT, \varphi \rangle = - \langle T, D\varphi \rangle \overset{(?)}{=} \langle S, \varphi \rangle$$

→ $\langle T, D\varphi \rangle = \langle T, \varphi' \rangle$

$= - \langle S, M(\varphi') \rangle$ since $\int_{-\infty}^{+\infty} \varphi' = 0$

$= - \langle S, \varphi \rangle$, by def. of M

Further, $|M(\varphi)|_{C^N(K)} \leq C\, |\varphi|_{C^N(K)}$

$|\langle S, M\varphi \rangle| \leq C\, |M(\varphi)|_{C^N(K)} \leq |\varphi|_{C^N}$

→ ■

- $f \in C^\infty$, $\varphi \in C_0^\infty$ $\Rightarrow$ $f\varphi \in C_0^\infty$.

- $(x-x_0) \in C^\infty$; ~~$(x-x_0)T \in C_0^\infty$~~

  $(x-x_0)T = 0 \Rightarrow T = c\,\delta_{x_0}$, c = constant.

- $(x-x_0)^N T = 0 \Rightarrow T = \sum_{i=0}^{N-1} c_i \delta^{(i)}$

- $f \in C^\infty$, $T \in \mathcal{D}'$, then

$$\boxed{D(fT) = f'T + fDT}$$

PRODUCT RULE APPLIES.

- Solve $\underbrace{xy' + y}_{\| \; (xy)'} = 0$

$(xy)' = 0 \Rightarrow xy = c$

$y = \frac{c}{x}$.

- $\underbrace{xT' + T}_{D(xT)} = 0$, $T \in \mathcal{D}'$

$D(xT) = 0 \Rightarrow xT = c$

$\Rightarrow T = c_1 \delta + c_2 \text{ P.V.}\left(\frac{1}{x}\right)$.

$\longrightarrow \infty$

... $x \cdot P.V.\left(\frac{1}{x}\right) = \int \varphi$.

$$\Rightarrow \quad x\Big(\underbrace{T - c\ P.V.\ \frac{1}{x}}_{=\delta}\Big) = 0$$

$$\Rightarrow \quad T = c_1 \delta + c_2\ P.V.\ \frac{1}{x} .$$

- Polynomial $= P(x)$.

  If $P(x)\,T = 0$, factorize and use previous properties.

- **TEMPERED DISTRIBUTIONS**: Distributions that behave nicely at ∞.

- $T \in \mathcal{D}'$ if $\forall\ K$, $\exists\ C_K$ and $N(K)$ s.t. $\forall\ \varphi \in C_0^\infty(K)$

$$|\langle T, \varphi \rangle| \leq C_K\ |\varphi|_{C^N(K)} .$$

With TEMPERED DISTRIBUTION $C_K = C$, $N(K) = N$ and just condition at ∞.

$\Rightarrow$ ...

- $T \in \mathcal{S}'$ if $\exists\, p$ s.t.

  $\forall\ \varphi \in C_0^\infty(\mathbb{R}^n)$

$$|\langle T, \varphi \rangle| \leq C\, N_p(\varphi),$$

where $N_p(\varphi) = \sup_x \sum_{\alpha+\beta \leq p} |x^\alpha \partial^\beta \varphi(x)|$