Clan #5 :

SOBOLEV SPACES : $1 \le p \le \infty$, $k \in \mathbb{Z}_+$.

- The SOBOLEV SPACE $W^{k,p}(\mathbb{R}^n)$ consists of all functions $u \in L^1_{loc}(\mathbb{R}^n)$ s.t. $D^\alpha u$ exists in the weak sense and belongs to $L^p(\mathbb{R}^n)$, $\forall\ |\alpha| \le k$.

- If $u \in W^{k,p}(\mathbb{R}^n)$, define its norm by

$$\|u\|_{W^{k,p}} := \begin{cases} \left(\sum_{|\alpha| \le k} \int |D^\alpha u|^p \, dx \right)^{1/p}, & (1 \le p < \infty) \\ \left(\sum_{|\alpha| \le k} \|D^\alpha u\|_\infty \right), & p = \infty. \end{cases}$$

- $u, \{u_m\}_{m=1}^\infty \subset W^{k,p}(\mathbb{R}^n)$. We say

$$u_m \xrightarrow{W^{k,p}} u$$

provided

$$\lim_{m \to \infty} \|u_m - u\|_{W^{k,p}} = 0.$$

• **THM**: For each $k = 1, 2, \ldots$ and $1 \le p \le \infty$ the Sobolev Space $W^{k,p}(\mathbb{R}^n)$ is a BANACH SPACE.

PROOF:

• $\|\cdot\|_{W^{k,p}}$ is a norm ✓

• Now let $\{u_m\}_{m=1}^{\infty} \subset W^{k,p}$ be Cauchy in $W^{k,p}$-norm. Then, for each $|\alpha| \le k$ $\{D^\alpha u_m\}_{m=1}^{\infty}$ is Cauchy in L^p, and since L^p is COMPLETE, $\exists\, u_\alpha \in L^p$ s.t.

$$D^\alpha u_m \xrightarrow{L^p} u_\alpha ;$$

In particular,

$$u_m \xrightarrow{L^p} u_{(0,0,\ldots,0)} =: u .$$

Now let $\phi \in C_c^\infty(\mathbb{R}^n)$. Then, since L^p convergence $\Rightarrow$ Weak convergence:

$$\langle D^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \phi \rangle$$

$$= (-1)^{|\alpha|} \lim_{m\to\infty} \langle u_m, D^\alpha \phi \rangle$$

$$= \underbrace{(-1)^{2|\alpha|}}_{=1} \lim_{m\to\infty} \langle D^\alpha u_m, \phi \rangle = \ldots$$

$$\dots = \langle u_\alpha, \phi \rangle$$

$$\Rightarrow \quad D^\alpha u = u_\alpha \quad \text{and} \quad u \in W^{k,p}(\mathbb{R}^n).$$

$$\Rightarrow \quad W^{k,p}(\mathbb{R}^n) = W^{k,p} \text{ IS COMPLETE } \blacksquare$$

- **THM**: $C_c^\infty(\mathbb{R}^n)$ is a DENSE subspace of $W^{k,p}(\mathbb{R}^n)$, for $1 \le p < \infty$.

PROOF: Clearly $C_c^\infty \subset W^{k,p}$.

Now take our usual mollifier $\varphi_\varepsilon(x)$ where $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right)$, $\varphi \in C_c^\infty(\mathbb{R}^n)$.

$\Rightarrow$ If $u \in W^{k,p}$, $u * \varphi_\varepsilon(x) \in C^\infty(\mathbb{R}^n)$.

Further, define $u_\varepsilon := u * \varphi_\varepsilon$, then

$$D^\alpha u_\varepsilon(x) = D^\alpha(u * \varphi_\varepsilon)(x) = \int u(y)\, D_x^\alpha \varphi_\varepsilon(x-y)\, dy$$

$$= (-1)^\alpha \int u(y)\, D_y^\alpha \varphi_\varepsilon(x-y)\, dy$$

$$= \int D^\alpha u(y)\, \varphi_\varepsilon(x-y)\, dy$$

$$= (D^\alpha u) * \varphi_\varepsilon(x), \quad \forall\, |\alpha| \le k.$$

... Thus, since $D^\alpha u \in L^p$ for $|\alpha| \le k$ we get that

$$D^\alpha u_\varepsilon \xrightarrow{L^p} D^\alpha u \quad , \quad \forall\, |\alpha| \in k.$$

$$\Rightarrow \quad \| u^\varepsilon - u \|_{W^{k,p}} \xrightarrow{\varepsilon \downarrow 0} 0$$

$$\Rightarrow \quad C^\infty(\mathbb{R}^n) \cap W^{k,p} \text{ is dense in } W^{k,p}.$$

For the compact case, consider $\rho \in C_c^\infty$ s.t. $\rho = 1$ in a neighborhood of zero.

$$\rightsquigarrow \quad \rho_\varepsilon(x) := \rho\left(\frac{\varepsilon x}{\blacksquare}\right) \quad \text{and consider}$$

$u \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$. Then,

$$u_\varepsilon(x) := u(x)\, \rho_\varepsilon(x) \in C_c^\infty(\mathbb{R}^n).$$

Further,

$$D u_\varepsilon(x) = \underbrace{\big(Du(x)\big)\rho_\varepsilon(x)}_{\downarrow (D.C.T.)\ L^p} + \underbrace{u(x)\, D\rho_\varepsilon(x)}_{\downarrow (D.C.T.)\ L^p}$$

$$Du \qquad\qquad 0$$

$$\Rightarrow C_c^\infty(\mathbb{R}^d) \text{ is DENSE in } C^\infty \cap W^{k,p} \qquad \blacksquare$$

- **COROLLARY**: $S(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$ for $1 \leq p < \infty$.

$$\left(\text{since } C_c^\infty \subset S \subset W^{k,p} \right)$$

- **REMARK**: $W^{1,2}(\mathbb{R}^n) = H^1(\mathbb{R}^n)$,

$$W^{k,2}(\mathbb{R}^n) = H^k(\mathbb{R}^n)$$

are all HILBERT SPACES with inner product defined as (for $H^1(\mathbb{R}^n)$):

$$\langle u, v \rangle_{H^1} := \int u \bar{v} + \nabla u \cdot \overline{\nabla v}$$

- For this section assume $1 \leq p < n$.
- Want to establish estimate of the form

$$\text{(*)} \qquad \|u\|_{L^q} \leq C \|Du\|_{L^p}$$

for $C > 0$, $1 \leq q < \infty$, and all functions $C_c^\infty(\mathbb{R}^n)$. C SHOULD NOT depend on u.

- **MOTIVATION:** If any inequality of the form (*) is possible, then it has to have a specific form.
Suppose $u \not\equiv 0$, $u \in C_c^\infty$.
Define $u_\lambda(x) = u(\lambda x)$,
$\lambda > 0$, $x \in \mathbb{R}^n$.

Suppose $\|u_\lambda\|_{L^q} \leq C \|u_\lambda\|_{L^p}$ $\rightsquigarrow$ ...

... • $\int |u_\lambda|^q dx = \frac{1}{\lambda^n} \int |u(y)|^q dy$

• $\int |Du_\lambda|^p dx = \lambda^p \int |Du(\lambda x)|^p dx$

$$= \frac{\lambda^p}{\lambda^n} \int |Du(y)|^p dy$$

$$\rightsquigarrow \quad \frac{1}{\lambda^{n/q}} \|u\|_{L^q} \leq C \frac{\lambda}{\lambda^{n/p}} \|u\|_{L^p}$$

$$\rightsquigarrow \quad \|u\|_{L^q} \leq C \lambda^{\left(1 - \frac{n}{p} + \frac{n}{q}\right)} \|u\|_{L^p}.$$

If $1 - \frac{n}{p} + \frac{n}{q} \neq 0$, can obtain contradiction by letting λ go to either 0 or ∞. Thus, if $\circledast$ holds need $1 - \frac{n}{p} + \frac{n}{q} = 0$, so that

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n},$$

$$q = \frac{np}{n-p}.$$

- **DEF**: If $1 \leq p < n$, the SOBOLEV CONJUGATE of p is

$$p^* = \frac{n\,p}{n-p}.$$

Note that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \quad p^* > p.$$

↳ We thus showed that the inequality (**) is possibly true only for $q = p^*$.

- **THM** : (Gagliardo-Nirenberg-Sobolev ineq.).

Assume $1 \leq p < n$. There exists a constant C, depending on p and n only, such that

$$\| u \|_{L^{p^*}} \leq C \| Du \|_{L^p},$$

for all $u \in C_c^\infty(\mathbb{R}^n)$.

NOTE: Need u to have COMPACT SUPPORT!! as the example $u \equiv 1$ shows.

PROOF: Start with $p = 1$, and suppose $n = 3$ (proofs for $n \geq 3$ are similar).

$$|u(x_1, x_2, x_3)| \leq \int_{-\infty}^{x_1} |\partial_1 u(s_1, x_2, x_3)| \, ds_1$$

$$\leq \int_{-\infty}^{+\infty} |Du(s_1, x_2, x_3)| \, ds_1 .$$

$\rightsquigarrow$

$\rightsquigarrow \quad \frac{n}{n-1} = \frac{3}{2}$, and by above

$$|u(x_1,x_2,x_3)|^{\frac{3}{2}} \leq \left(\int_{-\infty}^{+\infty} |Du(s_1,x_2,x_3)|\,ds_1\right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^{+\infty} |Du(x_1,s_2,x_3)|\,ds_2\right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^{+\infty} |Du(x_1,x_2,s_3)|\,ds_3\right)^{\frac{1}{2}}$$

Thus, integrate in x_1:

$$\int |u(x_1,x_2,x_3)|^{\frac{3}{2}}\,dx_1 \leq \left(\int_{-\infty}^{+\infty} |Du(s_1,x_2,x_3)|\,ds_1\right)^{\frac{1}{2}} \cdot \int_{-\infty}^{+\infty} \left\{\left(\int_{-\infty}^{+\infty} |Du(x_1,s_2,x_3)|\,ds_2\right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^{+\infty} |Du(x_1,x_2,s_3)|\,ds_3\right)^{\frac{1}{2}} dx_1$$

(HOLDER INEQ. with $p=2$) $\rightarrow \leq$

$$\left(\int_{-\infty}^{+\infty} |Du(s_1,x_2,x_3)|\,ds_1\right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |Du(x_1,s_2,x_3)|\,ds_2\,dx_1\right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |Du(x_1,x_2,s_3)|\,ds_3\,dx_1\right)^{\frac{1}{2}} \rightarrow$$

... Now integrate w.r.t. x_2 and use HÖLDER AGAIN:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |u(x_1,x_2,x_3)|^{\frac{3}{2}} dx_1 dx_2 \leq$$

$$\leq \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |Du(x_1,s_2,x_3)| ds_2 dx_1\right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |Du(s_1,x_2,x_3)| ds_1 dx_2\right)^{\frac{1}{2}}$$

$$\cdot \underbrace{\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |Du(x_1,x_2,s_3)| ds_3 dx_1 dx_2\right)^{\frac{1}{2}}}_{\text{by FUBINI } \|Du\|_{L^1}^{\frac{1}{2}}}$$

Integrate in x_3, again use Hölder to get

$$\| |u|^{\frac{3}{2}} \|_{L^1} \leq \|Du\|_{L^1}^{\frac{3}{2}}$$

$$\Rightarrow \|u\|_{L^{\frac{3}{2}}} \leq \|Du\|_{L^1}.$$

In general, one uses HÖLDER in the GENERALIZED form with exponent $p = n-1$ $\longrightarrow$

→ For $\mathbb{R}^n$ and $p = 1$

$$\boxed{\|u\|_{L^{\frac{n}{n-1}}} \leq \|Du\|_{L^1}.}$$

Now consider case $1 < p < n$; Apply previous estimate to $|u|^\gamma$:

$$\left(\int |u|^{\frac{\gamma n}{n-1}}\right)^{\frac{n-1}{n}} \leq \int |D(|u|^\gamma)|$$

$$= \gamma \int |u|^{\gamma-1} |Du|$$

$$\text{(HÖLDER)} \leq \gamma \left(\int |u|^{(\gamma-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int |Du|^p\right)^{\frac{1}{p}}$$

⇒ Choose γ s.t. $(\gamma-1)\frac{p}{p-1} = \frac{\gamma n}{n-1}$

$$\Rightarrow \quad \gamma = \frac{p(n-1)}{n-p} > 1.$$

$$\Rightarrow \frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1} = \frac{np}{n-p} = p^* \quad \longrightarrow \text{...}$$

→ Thus, the last estimate becomes

$$\|u\|_{L^{p^*}} \leq \frac{p(n-1)}{n-p} \|Du\|_{L^p}. \quad \blacksquare$$

• **COROLLARY**: Suppose $\frac{1}{r} = \frac{\theta}{p^*} + \frac{(1-\theta)}{q}$, $\theta \in (0,1)$ and $\theta = 1$ only if $r = \infty$. Then,

$$\|u\|_{L^r} \leq \|Du\|_{L^p}^{\theta} \|u\|_{L^q}^{(1-\theta)}$$

for any $u \in C_c^\infty(\mathbb{R}^n)$.

PROOF:

$$\int |u|^r = \int |u|^{r\theta} |u|^{r(1-\theta)}$$

$$\left(\begin{array}{c}\text{HÖLDER} \\ \text{since} \\ 1 = \frac{r\theta}{p^*} + \frac{r(1-\theta)}{q}\end{array}\right) \leq \left(\int |u|^{r\theta \frac{p^*}{r\theta}}\right)^{\frac{r\theta}{p^*}} \left(\int |u|^{r(1-\theta)\frac{q}{r(1-\theta)}}\right)^{\frac{r(1-\theta)}{q}}$$

$$= \|u\|_{L^{p^*}}^{r\theta} \|u\|_{L^q}^{r(1-\theta)}$$

$$\left(\begin{array}{c}\text{SOB.} \\ \text{INEQ.}\end{array}\right) \leq \left(\frac{p(n-1)}{n-p}\right)^{r\theta} \|Du\|_{L^p}^{r\theta} \|u\|_{L^q}^{r(1-\theta)} \longrightarrow$$

...

... Thus,

$$\|u\|_{L^n} \leq \left(\frac{p(n-1)}{n-p}\right)^{\theta} \|Du\|_{L^p}^{\theta} \|u\|_{L^q}^{1-\theta}$$

■

- **THM**: For $\frac{1}{r} = (1-\theta)\left(\frac{1}{q} - \frac{1}{n}\right) + \theta\left(\frac{1}{p} - \frac{m}{n}\right)$, $j < m$, $\frac{j}{m} < \theta < 1$, we have

$$\|u\|_{L^r} \leq \|D^j u\|_{L^q}^{1-\theta} \|D^m u\|_{L^p}^{\theta}$$

NOT USED IN THIS CLASS

- **REMARK**: The first Sobolev Inequality implies directly the following embedding:

$$W^{1,p} \subset L^{p^*}$$

• EX: $n=3$, $p=2$.

$$\|u\|_{L^6} \lesssim \|Du\|_{L^2}.$$

$\rightsquigarrow$ $u_n \in C_c^\infty$ and can control $\|Du_n\|_{L^2}$, then by FATOU'S LEMMA

$$\circledast \quad \int \liminf |u_n|^6 \le \liminf \|u_n\|_{L^6}^6 \lesssim \liminf \|Du_n\|_{L^2}$$

$$\le C$$

• Now suppose $u_n \xrightarrow{n\uparrow\infty}$ CONSTANT $\neq 0$

by $\circledast$, $\Longrightarrow\Longleftarrow$

Thus, u_n CANNOT converge to a non-zero constant if can control $\|Du_n\|_{L^2}$.

• If $\|Du_n\|_{L^2} \xrightarrow{n\uparrow\infty} 0$, then $\liminf u_n \xrightarrow{n\uparrow\infty} 0$ A.E

- $S(\mathbb{R}^n) = \left\{ \varphi \in C^\infty(\mathbb{R}^n) : N_k(\varphi) := \sup_x \sum_{|\alpha|+|\beta|\le k} \left| x^\alpha \partial^\beta \varphi(x) \right| < \infty \right\}$
$\forall\ k.$

→ Schwartz Class $\subset C^\infty$ and decays faster than any polynomial at ∞.

→ $C_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$

$\Rightarrow S' \subset D'$, $S' =$ TEMPERED DISTRIBUTIONS.

- EX: $T \in D'$ if $\forall\ B_R(0)$, $\forall\ \varphi \in C_c^\infty(B_R(0))$
$\exists\ N_R, C_R$ s.t.

$$|\langle T, \varphi \rangle| \le C |\varphi|_{C^N}$$

↳ If C_R grows at most POLYNOMIALLY in R, then $T \in S'$.

- **EX:** $f \in L^1_{loc}$, then $f \in D'$.

  If f grows at most polynomially then $f \in S'$. For example, e^x not in S' but in D'.

- **FOURIER TRANSFORM "$\mathcal{F}$" has a natural space: S'**

- $f \in S$, $\hat{f}(\xi) = \int f(x)\, e^{-2\pi i x \xi}\, dx$

  - $\widehat{\partial_k f} = 2\pi i \xi_k \hat{f}$
  - $\widehat{x_j f} = -2\pi i\, \partial_{\xi_j} \hat{f}$
  - $\left(x^\alpha \partial^\beta f\right)\hat{} = (2\pi i)^{\beta-\alpha}\, \partial^\alpha \xi^\beta \hat{f}$

$\Rightarrow$ $\mathcal{F}: S \longrightarrow S$.

- $f^\vee(x) = \int e^{2\pi i x \xi} f(\xi)\, d\xi$

- $(\hat{f})^{\vee} = f$ (REMEMBER PROOF.)

- $\mathcal{F}^{*} = \mathcal{F}^{-1}$ (w.r.t. $\langle \cdot, \cdot \rangle_{L^2}$).

- $\left(e^{-\pi a |x|^2}\right)^{\wedge} = \dfrac{e^{\frac{-\pi |\xi|^2}{a}}}{a^{n/2}}$, $(\xi, x \in \mathbb{R}^n)$

- PLANCHEREL: $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$.

- PROPERTIES of $\wedge$:

$\wedge : L^1 \to L^\infty$

$\wedge : L^2 \to L^2$ (By PLANCHEREL & DENSITY of S in L^2).

$\wedge : S' \to S'$ (By DUALITY, i.e., EXTENSION THM for DISTRIB.).

- H^s - SPACES on $\mathbb{R}^n$:

- $u \in H^1$: $\int |u|^2 + |Du|^2 \, dx < \infty$.

→ $\int |\hat{u}|^2 + |\xi|^2 |\hat{u}|^2 \, d\xi < \infty$

→ Can define H^s for any $s \in \mathbb{R}$:

- $u \in H^s$: $|u|_{H^s}^2 := \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi$ $(< \infty)$

→ If $s < 0$, then $u \in S'$

→ If $s \in \mathbb{Z}_+$, then $H^s = W^{k,2}$.

- **REMARKS:** $H^s(\mathbb{R}^m) = H^s$.

  - H^s, $s \geq 0$, space of functions.
  - H^s, $s < 0$, space of S'.
  - $H^k = W^{k,2}$ (PLANCHEREL).

- **TO REMEMBER:**

Polynomial $|P(\xi)| \leq C_1 + C_2 |\xi|^\ell$,

where $\deg(P) = \ell$.

and $C_1 |\xi|^\ell - C_2 \leq |P(\xi)|$

$\Rightarrow$ $C_1 |\xi|^\ell - C_2 \leq |P(\xi)| \leq C_1 + C_2 |\xi|^\ell$.

- H^s are all HILBERT SPACES because different s just change measure by factor of $(1+|\xi|^2)^s$.
- $(H^s)' = H^{-s}$, where $(')$ = Dual.
- $$|\langle u, \varphi \rangle| = \left| \int u \varphi \right|$$
$$= \left| \int \hat{u} \hat{\varphi} \right| \quad \left(\text{Since } \int \hat{f} g = \int f \hat{g} \right)$$
$$\leq \int \frac{1}{(1+|\xi|^2)^{s/2}} |\hat{u}| (1+|\xi|^2)^{s/2} |\hat{\varphi}| \, d\xi$$
$$\leq \underbrace{|u|_{H^{-s}}}_{C''} \cdot |\varphi|_{H^s}$$
$$= C'' \cdot |\varphi|_{H^s}$$

where $u \in H^{-s}$, $\varphi \in H^s$. Thus, conclude

$$(H^s)' = H^{-s}.$$

• **EX:**

• $\Delta u = 0$ on $\mathbb{R}^n$, u a classical solution and u bounded $\Rightarrow$ $u \equiv$ CONSTANT by LIOUVILLE'S THM.

• Suppose $\Delta u = 0$ in sense of TEMPERED DISTRIBUTIONS. Then, take $\mathcal{F}$ of it:

$$(\partial_{x_i}^2 u)^{\wedge} = -2\pi |\xi_i|^2 \hat{u}(\xi)$$

$$\Rightarrow (\Delta u)^{\wedge} = -(2\pi)^2 |\xi|^2 \hat{u}(\xi)$$

$\Rightarrow$ $\hat{u}$ is in $\mathcal{S}'$ & supported at $\{0\}$, thus

$$\hat{u} = \sum c_\alpha \delta^{(\alpha)}$$

$\Rightarrow$ $u =$ POLYNOMIAL, and if a polynomial is BOUNDED, then it is CONSTANT. ∎

(THIS is another proof of LIOUVILLE'S THM.)

- **Ex:**

$$(1-\Delta)u = f \;, \quad f \in L^2, \quad u \in S'.$$

$$\Downarrow$$

$$\hat{f} \in L^2.$$

Take $\mathcal{F}$:

$$(1+|\xi|^2)\,\hat{u} = \hat{f}(\xi)$$

$$\Rightarrow \quad \hat{u} = \frac{\hat{f}(\xi)}{1+|\xi|^2}$$

$$\Rightarrow \quad (\hat{u}(\xi))^{\vee} = \left(\frac{\hat{f}}{1+|\xi|^2}\right)^{\vee}$$

$$= u(x)$$

$$\Rightarrow \quad \boxed{u \in H^2(\mathbb{R}^n)}, \quad \text{since}$$

$$|u|_{H^2} = \int (1+|\xi|^2)^2 \,|\hat{u}(\xi)|^2 \, d\xi =$$

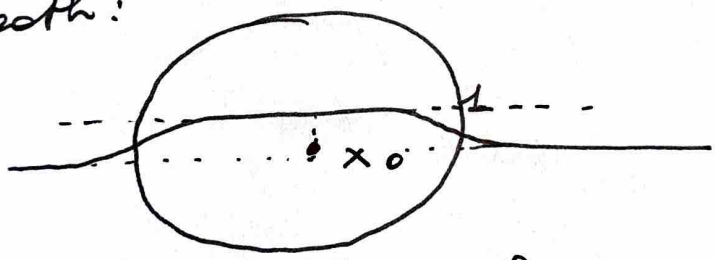
$$= \int |\hat{f}(\xi)|^2 \, d\xi = |f|_{L^2} \;\blacksquare$$

- What if want to talk about things on a local domain? Don't have F.T. on bdd. domain.

- **LOCALIZATION**:

$\Delta u = f$ on Ω, $f \in L^2$, $u \in H^1$.
Want to know $u \in H^2_{loc}$.

$\Rightarrow$ Take φ "CUTOFF FUNCTION" and smooth:



$\rightsquigarrow$ $\Delta(u\varphi) = f\varphi + 2\nabla u \cdot \nabla\varphi + u\Delta\varphi$

$\rightsquigarrow$ $(\Delta - 1)(u\varphi) = 2\nabla u \cdot \nabla\varphi + u\Delta\varphi + f\varphi - u\varphi$

$\Rightarrow$ $u\varphi \in H^2$ LOCALLY.