Class #6 SUPPLEMENT:

GLOBAL APPROXIMATION BY SMOOTH FUNCTIONS:

THM (1): $U \subset \mathbb{R}^n$, U bounded and $\partial U \in C^1$. Let $u \in W^{k,p}(U)$, $1 \leq p < \infty$. Then, $\exists \{u_m\} \subset C^\infty(\bar{U})$ s.t.

$$u_m \longrightarrow u \quad \text{in } W^{k,p}(U).$$

EXTENSION THM (2): $U \subset \mathbb{R}^n$, U bounded and $\partial U \in C^1$. Select any bounded open set V s.t. $U \subset\subset V$. Then, $\exists$ a BOUNDED LINEAR OPERATOR

$$E : W^{1,p}(U) \longrightarrow W^{1,p}(\mathbb{R}^n)$$

s.t. for any $u \in W^{1,p}(U)$:

(i): $Eu = u$ A.E. in U.

(ii): $\operatorname{supp}\{Eu\} \subset V$

(iii): $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C(p,U,V)\|u\|_{W^{1,p}(U)}$.

- **THM (3)**: (ESTIMATES for $W^{1,p}$, $1 \le p < n$).

Let U be a bounded, open subset of $\mathbb{R}^n$, $\partial U \in C^1$. Assume $1 \le p < n$ and $u \in W^{1,p}(U)$. Then, $u \in L^{p^*}(U)$ with estimate:

$$\| u \|_{L^{p^*}(U)} \le C(p, n, U) \| u \|_{W^{1,p}(U)}.$$

PROOF: Since ∂U is smooth, $\exists$ $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$ s.t. $\bar{u} = u$ on U, $\bar{u}$ has compact support, and

$$\| \bar{u} \|_{W^{1,p}(\mathbb{R}^n)} \le C \| u \|_{W^{1,p}(U)}.$$

- Since C_c^∞ is dense in $W^{1,p}(\mathbb{R}^n)$, $\exists$ $u_m \longrightarrow \bar{u}$ in $W^{1,p}(\mathbb{R}^n)$.
- By SOBOLEV INEQUALITY on $\mathbb{R}^n$,

$$\| u_m - u_\ell \|_{L^{p^*}(\mathbb{R}^n)} \le C \| D(u_m - u_\ell) \|_{L^p(\mathbb{R}^n)}$$

$\Longrightarrow \cdots$

... Thus, $\{u_m\}$ is Cauchy in $L^{p^*}(\mathbb{R}^n)$.

$\Rightarrow$ $u_m \to \bar{u}$ in $L^{p^*}(\mathbb{R}^n)$.

Again,

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \le C \|Du_m\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \le C \|D\bar{u}\|_{L^p(\mathbb{R}^n)}$$

$$\le C \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)}$$

$$\le \|u\|_{W^{1,p}(U)} \quad \left(\begin{matrix}\text{by}\\ \text{above.}\end{matrix}\right)$$

$\Rightarrow$ $u \in L^{p^*}(U)$ and

$$\|u\|_{L^{p^*}(U)} \le C \|u\|_{W^{1,p}(U)} \quad \blacksquare$$

- $W_0^{1,p}(\Omega)$ = Completion of $C_c^\infty(\Omega)$ under $W^{1,p}(\Omega)$-norm, where $\Omega \subseteq \mathbb{R}^n$.

- If $\Omega = \mathbb{R}^n$, we should that $C_c^\infty(\mathbb{R}^n)$ is DENSE in $W^{k,p}(\mathbb{R}^n)$, i.e.,

$$W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$$

- If Ω open and bounded subset of $\mathbb{R}^n$, then $W_0^{k,p}(\Omega) \subset W^{k,p}(\Omega)$, and from (THM ①), $W_0^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}^{W^{k,p}}$.

- **THM ④**: (ESTIMATE for $W_0^{1,p}$, $1 \le p < n$).
Assume U is bounded, open in $\mathbb{R}^n$.
Suppose $u \in W_0^{1,p}(U)$, $1 \le p < n$. Then,

$$\|u\|_{L^q(U)} \le C \|Du\|_{L^p(U)}$$

for each $q \in [1, p^*]$, (C is independent of u).

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- **REMARK:** For $q = p$ (allowed since $p^* > p$), we have

$$\|u\|_{L^p(U)} \leq \|Du\|_{L^p(U)}$$

$\Rightarrow$ In view of this estimate, on $W_0^{1,p}(U)$ the norms

$$\boxed{\|Du\|_{L^p(U)} \sim \|u\|_{W^{1,p}(U)}},$$

if U is bounded.

PROOF: $u \in W_0^{1,p}(U)$, then $\exists$ $\{u_m\} \subset C_c^\infty(U)$ s.t.

$$u_m \longrightarrow u \text{ in } W^{1,p}(U).$$

$\rightsquigarrow$ Extend $u_m = 0$ on $\mathbb{R}^n \setminus U$ and apply SOBOLEV ineq.:

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq c \|Du_m\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c \|Du\|_{L^p(\mathbb{R}^n)}.$$

Since $|U| < \infty$, then

$$\|u\|_{L^q(U)} \leq c \|u\|_{L^{p^*}(U)}, \quad q \in [1, p^*]$$

- We used SOBOLEV INEQUALITY on $\mathbb{R}^n$ to show that

$$W^{1,p}(U) \subset L^{p^*}(U)$$

for $1 \leq p < n$.

- Now we show that

$$W^{1,p}(U) \subset\subset L^q(U)$$

for $1 \leq q < p^*$.

- **DEF**: X, Y Banach Spaces, $X \subset Y$.
We say X is COMPACTLY EMBEDDED in Y, written

$$X \subset\subset Y,$$

provided

(i): $\|u\|_Y \leq C \|u\|_X$, for any $u \in X$

(ii): Each BOUNDED SEQUENCE in X is PRECOMPACT in Y.

Condition (ii) means that for $\{u_k\} \subset X$ s.t.

$$\sup_k \|u_k\|_X < \infty$$

then $\exists$ $\{u_{k_j}\}$ SUBSEQUENCE s.t.

$$\lim_{j \to \infty} \|u_{k_j} - u\|_Y,$$

i.e., u_{k_j} converges to some $u \in Y$ as $j \uparrow \infty$.

- **THM** ⑤ (RELLICH-COMPACTNESS)

Assume U is a bdd., open subset of $\mathbb{R}^n$ and $\partial U \in C^1$. Suppose $1 \leq p < n$. Then,

$$\boxed{W^{1,p}(U) \subset\subset L^q(U)},$$

for each $1 \leq q < p^*$.

PROOF: Fix $1 \le q < p^*$. Since U is bdd, by (THM ③) we see that

$$W^{1,p}(U) \subset L^q(U), \quad \text{i.e.,}$$

$$\|u\|_{L^q(U)} \le C \|u\|_{W^{1,p}(U)}.$$

$\rightsquigarrow$ Need to show that if $\{u_m\}_1^\infty$ is a BDD. sequence in $W^{1,p}(U)$, $\exists$ subsequence $\{u_{m_j}\}_{j=1}^\infty$ which converges in $L^q(U)$.

- By EXTENSION THM, assume $U = \mathbb{R}^n$ and $\{u_m\}_{m=1}^\infty$ all have COMPACT SUPPORT in some BDD. $V \subset \mathbb{R}^n$.

  Assume

$$\sup_m \|u_m\|_{W^{1,p}(V)} < \infty.$$

- φ_ε our usual mollifiers, $\varphi \in C_c^\infty(\mathbb{R}^n)$

$\rightsquigarrow$ $u_m^\varepsilon := \varphi_\varepsilon * u_m$ have all support inside V, bdd $\subset \mathbb{R}^n$ (w.l.g.)

ooo ⇝ (i):

$$u_m^\varepsilon := \eta_\varepsilon * u_m$$

$$\downarrow \ \varepsilon \downarrow 0$$

$$u_m$$

UNIFORMLY in m.

PROOF: For u_m smooth:

$$u_m^\varepsilon(x) - u_m(x) = \int_{B(x,\varepsilon)} \varphi(z)\left[u_m(x-\varepsilon z) - u_m(x)\right] dz$$

$$= \int_{B(0,1)} \varphi(z) \int_0^1 \frac{d}{dt} u_m(x-\varepsilon t z)\, dt\, dz$$

$$= -\varepsilon \int_{B(0,1)} \varphi(z) \int_0^1 Du_m(x-\varepsilon t z)\cdot z \, dt\, dz$$

$$\Rightarrow \int_V \left| u_m^\varepsilon(x) - u_m(x)\right| dx \le$$

(FUBINI) $\le \varepsilon \int_{B(0,1)} |\varphi(z)| \int_0^1 \int_V |Du_m(x-\varepsilon t z)| \underbrace{|z|}_{\le 1} dx\, dt\, dz$ (CAUCHY-SCHWARTZ:)

$$\le \varepsilon \underbrace{\int |\varphi(z)| dz}_{\| \ 1} \int_V |Du_m(y)|\, dy$$

(By TRANSLATION INVARIANCE).

By DENSITY, we conclude for $u_m \in W^{1,p}(V)$:

$$\| u_m^\varepsilon - u_m \|_{L^1(V)} \leq \varepsilon \| Du_m \|_{L^1(V)}$$

$$\leq \varepsilon C \| Du_m \|_{L^p(V)}^{(<\infty)},$$

since V is BDD. Thus,

$$u_m^\varepsilon \xrightarrow{\varepsilon \downarrow 0} u_m \quad \left(\text{UNIFORMLY in } m\right) \text{ in } L^1(V)$$

But then, by INTERP. HÖLDER with $1 \leq q < p^*$ and $\frac{1}{q} = \theta + \frac{(1-\theta)}{p^*}$:

$$\| u_m^\varepsilon - u_m \|_{L^q(V)} \leq \| u_m^\varepsilon - u_m \|_{L^1(V)}^{\theta} \| u_m^\varepsilon - u_m \|_{L^{p^*}(V)}^{1-\theta}$$

By SOBOLEV INEQ :

$$\| u_m^\varepsilon - u_m \|_{L^q(V)} \leq C \varepsilon \| u_m^\varepsilon - u_m \|_{L^1(V)}^{\theta}.$$

Thus, (i) is PROVED.

(ii): For each fixed $\varepsilon > 0$, $\{u_m^\varepsilon\}_{m=1}^\infty$ is UNIFORMLY BDD. & EQUICONTINUOUS.

PROOF:

$$|u_m^\varepsilon(x)| \le \int_{B(x,\varepsilon)} \varphi_\varepsilon(x-y)\,|u_m(y)|\,dy$$

$$\le \|\varphi_\varepsilon\|_{L^\infty}\,\|u_m\|_{L^1(V)}$$

$$\le \frac{C}{\varepsilon^n} < \infty \qquad (\text{HÖLDER})$$

$$|Du_m^\varepsilon(x)| \le \int_{B(x,\varepsilon)} |D\varphi_\varepsilon(x-y)|\,|u_m(y)|\,dy$$

$$\le \frac{C}{\varepsilon^{n+1}} < \infty \qquad (\text{HÖLDER}).$$

$$\Rightarrow |u_m^\varepsilon(x) - u_m^\varepsilon(y)| = \left|\int_0^1 \frac{d}{dt}\, u_m^\varepsilon(tx+(1-t)y)\,dt\right|$$

$$\le \int_0^1 |Du_m^\varepsilon(tx+(1-t)y)|\,|x-y|\,dt$$

$$\le \frac{C}{\varepsilon^{n+1}}|x-y| \qquad \text{and (ii) is PROVED.}$$

(iii): Fix $\delta > 0$. Then, $\exists$ subsequence $\{u_{m_j}\}_{j=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ s.t.

$$\limsup_{j,k \to \infty} \| u_{m_j} - u_{m_k} \|_{L^q(V)} \leq \delta.$$

PROOF: • First choose $\varepsilon > 0$ so small so that

$$\| u_m^\varepsilon - u_m \|_{L^q(V)} \leq \frac{\delta}{2}.$$

• Now since $\{u_m\}$ have support in V bounded, then $\{u_m^\varepsilon\}$ have also support in V bounded. Thus, By ARZELA-ASCOLI COMPACTNESS CRITERION, $\exists \{u_{m_j}^\varepsilon\} \subset \{u_m^\varepsilon\}$ s.t. $\{u_{m_j}^\varepsilon\}$ CONVERGES UNIFORMLY on V. Thus,

$$\limsup_{j,k \to \infty} \| u_{m_j}^\varepsilon - u_{m_k}^\varepsilon \|_{L^q(V)} \overset{(D.C.T.)}{=} 0$$

$\Longrightarrow$ ...

$$\| u_{m_j} - u_{m_k} \|_{L^q(V)} \leq$$

$$\leq \| u_{m_j} - u_{m_j}^{\varepsilon} \|_{L^q(V)} + \| u_{m_j}^{\varepsilon} - u_{m_k}^{\varepsilon} \|_{L^q(V)} +$$

$$+ \| u_{m_k}^{\varepsilon} - u_{m_k} \|_{L^q(V)}$$

$$\leq \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta, \text{ and (iii) IS PROVED.}$$

(iv): Now Repeat assertion (iii) with $\delta_k = \frac{1}{k}$, $k \in \mathbb{Z}_+$ and use CANTOR DIAG. ARGUMENT to extract a subsequence

$$\{u_{m_\ell}\}_{\ell=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$$

s.t.

$$\limsup_{\ell,k \to \infty} \| u_{m_\ell} - u_{m_k} \|_{L^q(V)} = 0$$

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- **THM 6** (POINCARÉ INEQ.)

U a bdd., connected, open subset of $\mathbb{R}^n$, $\partial U \in C^1$. Assume $1 \le p < n$. Then, $\exists\ C(n,p,U)$ s.t.

$$\| u - u_{av} \|_{L^p(U)} \le C \| Du \|_{L^p(U)}$$

for each $u \in W^{1,p}(U)$. $\left[u_{av} = \fint_U u \, dy \right]$

PROOF: Suppose NOT. Then, for each $k = 1, 2, \ldots$, $\exists\ u_k \in W^{1,p}(U)$ s.t.

$$\| u_k - (u_k)_{av} \|_{L^p(U)} > k \| Du_k \|_{L^p(U)}.$$

Normalize by taking

$$v_k = \frac{u_k - (u_k)_{av}}{\| u_k - (u_k)_{av} \|_{L^p(U)}}$$

$$\Rightarrow \quad \| v_k \|_{L^p(U)} = 1, \qquad (v_k)_{av} = 0 \qquad \ldots$$

$$\rightsquigarrow \quad \| Dv_k \|_{L^p(U)} < \frac{1}{k}, \quad k = 1, 2, \ldots$$

$$\Rightarrow \quad \{v_k\}_{k=1}^{\infty} \text{ are bdd in } W^{1,p}(U)$$

and by RELLICH COMPACTNESS,

$\exists \ \{v_{k_j}\}_{j=1}^{\infty}$ and a $v \in L^p(U)$

s.t. $v_{k_j} \longrightarrow v$ in $L^p(U)$.

$\phi \in C_c^\infty(U)$:

$$\Rightarrow \left| \langle v, D_{x_i} \phi \rangle \right| = \lim_{j \to \infty} \left| \langle v_{k_j}, D_{x_i} \phi \rangle \right|$$

$$= \left| - \lim_{j \to \infty} \langle D_{x_i} v_{k_j}, \phi \rangle \right|$$

$$\leq C \cdot \lim_{j \to \infty} \left(\int_U |D_{x_i} v_{k_j}|^p \right)^{\frac{1}{p}}$$

$$\leq C \lim_{j \to \infty} \| Dv_k \|_{L^p(U)} = 0$$

$$\Rightarrow \quad v \in W^{1,p}(U) \quad \text{with} \quad Dv = 0 \ \text{A.E.}$$

$$\Rightarrow \quad v \equiv \text{constant}, \quad \text{with} \quad v_{av} = 0$$

$$\Rightarrow \quad v \equiv 0. \quad \text{But we had} \quad \|v\|_{L^p(U)} = 1$$

$$\Rightarrow \Leftarrow \quad \blacksquare$$

• **THM: (7)** (POINCARÉ'S INEQ. FOR BALL).

Assume $1 \le p < n$. Then, $\exists\ C(n,p)$ s.t.

$$\| u - (u)_{x,r} \|_{L^p(B(x,r))} \le C\, r\, \| Du \|_{L^p(B(x,r))}$$

for each $B(x,r)$ and each $u \in W^{1,p}(B(x,r))$.

$$\left[(u)_{x,r} = \fint_{B(x,r)} u\, dy \right].$$

PROOF: If $U = B(0,1)$, then this follows from (THM (6)).

In general, if $u \in W^{1,p}(B(x,r))$,

$$v(y) := u(x + ry), \quad y \in B(0,1).$$

$$\Rightarrow \quad v \in W^{1,p}(B(0,1)) \quad \text{and}$$

$$\| v - (v)_{0,1} \|_{L^p(B(0,1))} \le C \| Dv \|_{L^p(B(0,1))},$$

change variables and conclude ■

- **REMARK** : • Since $p^* > p$ and $p^* \to \infty$ as $p \to n$, then

$$\boxed{W^{1,p}(U) \subset\subset L^p(U)}$$

for all $1 \leq p \leq \infty$, since U is BOUNDED $\subset \mathbb{R}^n$ (By HÖLDER).

• Thus, ALSO POINCARE'S INEQ. hold for $1 \leq p \leq \infty$!!