Class # 6 :

- Poincaré Inequality
- $H_0^1(\Omega)$
- Rellich Compactness
- Elliptic Regularity - Lax Milgram.

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} \quad , \quad p < n$$

$$|u|_{L^{p^*}} \leq |Du|_{L^p} \longleftarrow \left(\left(\frac{1}{L} \right)^p L^n \right)^{1/p}$$

DIMENSIONAL ANALYSIS :

$$L^{\frac{n}{p^*}} \qquad L^{\frac{n}{p} - 1}$$

$$\Rightarrow \boxed{\frac{n}{p^*} = \frac{n}{p} - 1}$$

↳ This inequality holds also for C^∞ functions (not compactly supported, if I exclude $n = 1$).

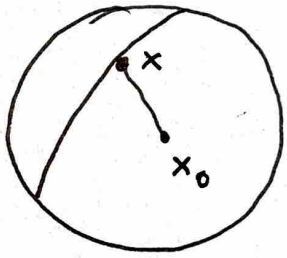
- Want similar Inequality but for $L^p(B_R(0))$ & $L^{p\blacklozenge}(B_R(0))$:

$$|u - u_{av}|_{L^{p\blacklozenge}(B_R(0))} \leq |\nabla u|_{L^p(B_R(0))}$$

$$\boxed{|u - u_{av}|_{L^{p\blacklozenge}(B_R(0))} \leq R^{\alpha} |\nabla u|_{L^p(B_R(0))}}$$

POINCARÉ' INEQUALITY

PROOF:



$$u(x) - u(x_0) = \int_0^r \frac{\partial u}{\partial r}(r', \omega)\, dr'$$

$$\Rightarrow \int_\omega |u(x) - u(x_0)|\, d\omega \leq \int_\omega \int_0^r \frac{\partial u}{\partial r}(r', \omega)\, dr\, d\omega$$

$$\Rightarrow \iint |u(x) - u(x_0)|\, d\omega\, r^{n-1} dr \leq$$

$$\leq \int \int_0^r r^{n-1} \left|\frac{\partial u}{\partial r}(r', \omega)\right| dr' dr$$

... ■

- **CONSEQUENCE OF POINCARÉ :**

(i): $\tilde{H} = \left\{ u \in H^1(\Omega) : \int_\Omega u = 0 \right\}$.

$\tilde{H}$ is an Hilbert Space with norm

$$|u|_{\tilde{H}^1} = \left(\int |\nabla u|^2 \right)^{1/2},$$ equivalent

to $$|u|_{H^1} = \left(\int |\nabla u|^2 + |u|^2 \right)^{1/2}$$

since by POINCARÉ' $|\nabla u|_{L^p}$ dominates $|u|_{L^p}$ since u have zero mean.

(For bdd. set $L^{p^*}(\Omega) \subset L^p(\Omega)$, since $p^* > p$).

(ii): **BOUNDARY VALUES of a FUNCTION :**

- $f \in L^2(\mathbb{R})$, $f(0)$ doesn't make any sense.
- $f \in H^1(\mathbb{R})$, $f(0)$.

↳ $\int |f(x)|^2 + |f'(x)|^2 \, dx < \infty$.

$$|f(x_1) - f(x_2)| = \left| \int_{x_2}^{x_1} f'(t) \, dt \right|$$

so f is continuous.

- $f \in H^1(\mathbb{R}^2)$

↳ In this case $p^* = \infty$, SOBOLEV DOESN'T HOLD

$\Rightarrow$ $f(0,0)$ DOESN'T MAKE ANY SENSE !!

BUT, can make sense of

$$f^2(x_1,0) = \int_{-\infty}^{0} \partial_2 f^2(x_1,s)\, ds$$

$$\Rightarrow \quad = \int_{-\infty}^{0} 2 f(x_1,s)\, \partial_2 f(x,s)\, ds$$

$$\int f^2(x_1,0)\, dx_1 = \iint_{-\infty}^{0} \partial_2 f^2(x_1,s)\, ds\, dx_1$$

$\Rightarrow$ Can make sense $< \infty$ of $f(x_1,0)$ as an L^2 function.

- **THM**: $f(x,y) \in H^s(\mathbb{R}^m \times \mathbb{R}^m)$ $\quad (W^{s,p})$

$$f(x,0) \in H^{s-\frac{m}{2}}(\mathbb{R}^m), \quad s > \frac{m}{2}.$$

$$\left(W^{s-\frac{m}{p}}(\mathbb{R}^m),\ s > \frac{m}{p}\right)$$

- $H_0^1(\Omega) = \{ u \in H^1(\Omega) \mid u|_{\partial\Omega} = 0 \}$

$\downarrow$

u on boundary $\in L^2$ and on $\partial\Omega = 0$ in L^2-sense.

↳ Since $u \in H_0^1(\Omega)$, can extend $u = 0$ outside of Ω

$$\tilde{u} = \begin{cases} u, & x \in \Omega \\ 0, & x \notin \Omega \end{cases}$$

This is fine since $u|_{\partial\Omega} = 0$ and difference quotient $D_k u$ still same at $\partial\Omega$.

$\Rightarrow \tilde{u} \in H^1(\mathbb{R}^n)$.

- $H_0^1(\Omega) = \{ \varphi \in C_0^\infty(\Omega) \mid \int |\nabla\varphi|^2 + |\varphi|^2 < \infty \}$

is another characterization.

$\rightsquigarrow u(x_1, x_2, \dots) = \int_{-\infty}^{x} \partial_1 u(s, x_2 \dots)\, ds$

$\Rightarrow$

$$\Rightarrow \quad \int u^2 \leq C \int |Du|^2 , \quad u \in H_0^1 .$$

$$H_0^1(\Omega) = \{ u \in H^1(\Omega) \mid u|_{\partial\Omega} = 0 ,$$

EQUIVALENT NORM. $\rightarrow |u|_{H_0^1} = \int |Du|^2 \}$

- **RELLICH COMPACTNESS** :

- **ARZOLA - ASCOLI** : BOUNDEDNESS + (UNIFORM) EQUICONTINUITY

  $\Rightarrow$ COMPACTNESS .

  (NEED COMPACTNESS OF DOMAIN) .

$\rightarrow$ Gain Equicontinuity from any kind of regularity (Hölder cont., Lipschitz ...)

REGULARITY $\Rightarrow$ EQUICONTINUITY

- $\boxed{\text{ARZELA ASCOLI FOR } L^p\text{-SPACES}}$:

- $L^p(\Omega)$, $|\Omega| < \infty$, $u_n \in L^p$ bounded and equicontinuous. Then, $\exists$ a subsequence $u_{n_k} \xrightarrow{L^p} u_*$.

- $\boxed{\text{IMPLICATION :}}$

$u_n \in W^{1,p}(\Omega)$, $\quad |u_n|_{W^{1,p}(\Omega)} \lesssim 1$.

$\rightsquigarrow \quad \int_\Omega |u_n(x+h) - u_n(x)|^p dx \leq c|h|^p$, $\quad \boxed{\begin{array}{l} p \neq 0 \\ 0 < p < 1 \\ 1 < p < \infty \end{array}}$

Because $\quad Du_n \overset{L^p}{=} \lim_{h \to 0} D_h u_n$ (HW PROBL.)

$\rightsquigarrow$ By ARZ. ASCOLI FOR L^p-SPACES, $\exists$ subsequence $u_{n_k} \longrightarrow u_*$ in L^p, and $u_* \in W^{1,p}$ by WEAK CONVERGENCE

- $W^{1,p}(\Omega) \overset{\text{"}\subset\text{"}}{\hookrightarrow} L^p(\Omega)$ ← By previous Thm, this EMBED IS COMPACT !!

- $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ $\quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} > 0,$

This Embedding is NOT COMPACT, but

$$W^{1,p}(\Omega) \subset\subset L^r(\Omega), \quad \forall\ r < p^*$$

• ELLIPTIC REGULARITY :

DIRICHLET PROBLEM
$$\begin{cases} -\Delta u = f, & \text{on } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}, \quad f \in L^2.$$

Assume $\exists$ solution $u \in H_0^1(\Omega)$.

NEUMANN PROBLEM
$$\begin{cases} -\Delta u = f, & f \in L^2, \\ n \cdot \nabla u|_{\partial\Omega} = 0 \end{cases}$$

Cannot talk about $u \in H_0^1(\Omega)$!!
Because boundary value involves derivatives.

• $\boxed{\text{LAX - MILGRIN}}$: $(H, \langle \cdot, \cdot \rangle)$.

H an Hilbert Space, $B : H \times H \longrightarrow \mathbb{R}$ a bilinear form. Suppose

$$|B(u,v)| \lesssim |u|_H \, |v|_H .$$

and $B(u,u) \gtrsim |u|^2$. Can I solve

$$\boxed{B(u,v) = \langle f, v \rangle , \quad f \in H'.} \quad ??$$

$f \in H'$, $\exists\, u \in H$ s.t. $B(u,v) = \langle f, v \rangle$, $\forall\, v \in H$.

Yes by <u>LAX - MILGRIN</u>.

<u>PROOF:</u> By RIESTZ REPRESENTATION:

$$B(u,v) = (Au, v).$$

⇝ Need to show can solve $(Au, v) = \langle f, v \rangle$

⇝ If A is invertible, done.

$$|u|\,|Au| \geq (Au, u) \geq C\, |u|^2$$

$$\Rightarrow \quad M\, |u|_H \geq |Au|_H \geq C\, |u|_H \quad \Rightarrow \cdots$$

... A is 1-1, and $R(A)$ is CLOSED

$\Rightarrow$ A is 1-1 and ONTO

$\Rightarrow$ A is invertible. ■

- **DIRICHLET PROBLEM**:

$$\int_\Omega -\Delta u = \int_\Omega f v$$

use integration by parts

$$\Rightarrow \int \nabla u \cdot \nabla v = \int f v \quad , \forall\, v \in H_0^1 .$$

$$\Rightarrow B(u,v) = \int_\Omega \nabla u \cdot \nabla v \quad , \text{ this satisfies}$$

2 inequalities of LAX-MILGRAM

$\Rightarrow$ $\exists$ solution, boundary conditions are incorporated in fact that $u \in H_0^1$.

If $u|_{\partial\Omega} = g$, consider $\varphi \in H^2(\Omega)$,

$\varphi|_{\partial\Omega} = g$, $w = u - \varphi \Rightarrow -\Delta w = f - \Delta\varphi$

and apply L-M.

- $Lu = -\sum \partial_i \left(a_{ij}(x)\, \partial_j u \right)$

(a_{ij}) matrix, entries are C^∞,

and it is POSITIVE DEFINITE, i.e.,

$$\sum_{ij} a_{ij}(x)\, \xi^i \xi^j \geq c\, |\xi|^2, \quad c > 0.$$

and symmetric, $a_{ij} = a_{ji}$.

$$\Rightarrow \begin{cases} Lu = f \\ u|_{\partial\Omega} = 0 \end{cases} \qquad u \in H^1_0(\Omega)$$

$\Rightarrow$ Integrate by parts,

$$B(u,v) = \int_\Omega a_{ij}\, \partial_i u\, \partial_j v,$$

don't get boundary terms since $u \in H^1_0(\Omega)$.

$$\Rightarrow \left| \sum_{ij} \int a_{ij}\, \partial_i u\, \partial_j v \right| \leq C(a) \int_\Omega |\nabla u|\, |\nabla v| \leq C\, |\nabla u|_{L^2}\, |\nabla v|_{L^2}$$

$\Rightarrow$ Apply L-M to get Solution. $|u|_{H^1_0}\, |v|_{H^1_0}$.

$$\sum_{ij} a_{ij}(x)\,\xi^i \xi^j \geq c\,|\xi|^2$$

is the Characteristic Equation of PDE $\Rightarrow$ ONLY way for LHS to be $=0$ is $|\xi|=0$

$\Rightarrow$ Eqn is ELLIPTIC (& LINEAR).

- **NEUMANN'S PROBLEM**:

$$\int_\Omega -\Delta u\, v = \int_\Omega f v.$$

Integrate by parts:

$$-\underbrace{\int_{\partial\Omega} n \cdot \nabla u\, v}_{=0} + \int_\Omega \nabla u \cdot \nabla v = \int f v.$$

$$\Rightarrow \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v$$

$$\Rightarrow B(u,v) := \int_\Omega \nabla u \cdot \nabla v = \int f v.$$

↓

Does this satisfy L-M ??

... POINCARRÉ TELLS ME

$$\int |w - w_{av}|^2 \, dx \leq \int |\nabla w|^2$$

Lax-Milgram FAILS in $\underline{\underline{H_1}}$, because

$$\int |\nabla w|^2 \not\geq c \int |w|^2 .$$

Need AVERAGE$(w) = 0$ so that POINCARRÉ' is satisfied & $B(w,w)$ satisfies L-M. $\Rightarrow$ Consider $\tilde{H}$

(Constant comes from non-uniqueness of NEUMANN'S PROBLEM.).

$\rightarrow$ PROBLEM IS SOLVED IN $\tilde{H}$.

How can I know that $n \cdot \nabla u|_{\partial\Omega} = 0$??

$\rightarrow$ VERIFY:

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v .$$

$\rightarrow$

$$\int_{\partial\Omega} (n \cdot \nabla u) v - \int (\Delta u) v = \int f v$$

$\rightarrow$ for $v \in C_0^\infty$