• Class # 7 :

(1): CACCIOPPOLI INEQUALITY

(2): INTERIOR REGULARITY

(3): BOUNDARY REGULARITY

→ Don't Tell you how to construct Solutions, Just theoretical facts.

⇓

(4): POTENTIAL THEORY.

• $-\Delta u = f$ on Ω, $f \in L^2$, $u \in H_0^1(\Omega)$.

$$u|_{\partial\Omega} = 0$$

$$\int_\Omega Du \cdot Dv = \int fv \longleftarrow \exists \text{ solution } u$$

by Riesz Representation (=Lax-Milgram in this case)

• Lax-Milgram more powerful as it tells me can solve equations of the form

$$\circledast \int_\Omega a^{ij} \partial_i u \, \partial_j v = \int_\Omega fv \qquad \left(-\partial_i(a_{ij}\partial_i u) = f \text{ INT. BY PARTS.}\right)$$

All on $\Omega \subset \mathbb{R}^n$:

- If $u \in H_0^1$, then $\Delta u \in H^{-1} = (H_0^1)^*$:

$u \in H_0^1(\Omega) \Rightarrow u \in H^1(\mathbb{R}^n)$ by extending it to $=0$ on $\mathbb{R}^n$

$\rightsquigarrow (H^1(\mathbb{R}^n))^* = (H^{-1}(\mathbb{R}^n))^*$

and we know $\Delta : H^1 \longrightarrow H^{-1}$ by F.T.

- If $-\Delta u = f$, $u \in H^1$, $v \in H^1$, $-\Delta u \notin (H^1)^*$

PROBLEM

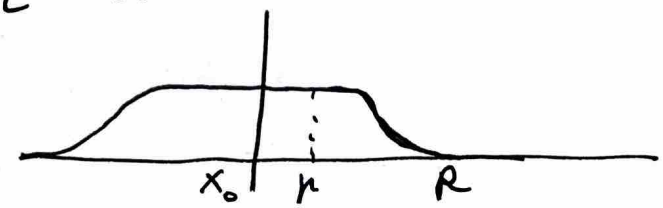
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(1): **CACIOPPOLI INEQUALITY** : Derived for POISSON PROBLEM BUT SAME DERIVATION FOR (*), i.e. POSITIVE DEFINITE, BOUNDED.

$u \in H_0^1(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad , \quad \forall v \in H_0^1 \quad \leftarrow \exists \text{ such } u \text{ by LAX-MILGREN.}$$

$v = \eta^2 u$, η CUTOFF FUNCTION



... Plug in $v = \eta^2 u$:

$$\to \int \eta^2 \nabla u \cdot \nabla u + 2\eta \underbrace{\nabla \eta}_{L^2} \cdot \nabla u \underbrace{u}_{L^2} = \int f \eta^2 u$$

(BOUND FROM BELOW)

$$\to \int \eta^2 |\nabla u|^2 - 2 |\eta \nabla u|_{L^2} |\nabla \eta u|_{L^2}$$

$$\leq |\eta f|_{L^2} |\eta u|_{L^2}$$

$$\to |\eta \nabla u|_{L^2}^2 - 2 |\eta \nabla u|_{L^2} |\nabla \eta u|_{L^2} \leq |\eta f|_{L^2} |\eta u|_{L^2}$$

(use Young's ineq. and $a^2+b^2 \leq (a+b)^2$, $a,b \geq 0$.):

$$\to |\eta \nabla u|_{L^2}^2 - 2\left(\frac{1}{4} |\eta \nabla u|_{L^2}^2 + 4 |(\nabla \eta) u|_{L^2}\right)$$

$$\leq |\eta f|_{L^2}^2 + |\eta u|_{L^2}^2$$

$$\to \frac{1}{2} |\eta \nabla u|_{L^2}^2 \leq C\left(|\nabla \eta u|_{L^2} + |\eta f|_{L^2} + |\eta u|_{L^2}\right)$$

$$\Rightarrow \boxed{\int_{B_r(x_0)} |\nabla u|^2 \leq C(R,r) \int_{B_R(x_0)} |u|^2 + C \int |f|^2}$$

CACCIOPPOLI → With this, can show that $u \in H^2_{loc}(\Omega)$.

- $u \in H_0^1(\Omega)$

$$-\Delta u = f \in L^2.$$

- **THM** : $u \in H^2_{loc}(\Omega)$.

PROOF : (Multiply by $\partial_i \partial_j \eta^2 u$) (*)

$$\int -\Delta u \, \partial_i \partial_j \eta^2 u = \int f \partial_i \partial_j \eta^2 u$$

INTEGRATE BY PARTS :

$$Y = \int \eta^2 |\nabla \partial_i u|^2 + \dots = \int f \partial_i \partial_j \eta^2 u$$

HÖLDER

$$\Rightarrow \quad Y^2 - C_1 Y \leq C_2 Y$$

$$\Rightarrow \quad Y \leq \frac{C_2}{1 + C_2}, \quad C_2 = |f|_{L^2}.$$

(*) : Cannot DO THIS, as $\partial_j \eta^2 u \notin H_0^1(\Omega)$

(REMEMBER : Can bound series of derivatives by H.O. & L.O. terms)

→ Can FIX this proof by convolving $u * \varphi_\varepsilon$, approximate identity.

$\Rightarrow$ $\eta^2 u_\varepsilon \in C^\infty$ and so $\eta^2 u_\varepsilon \in H_0^1(\Omega)$ ✓

→ Will do it in ANOTHER WAY:

$$\int \nabla u \cdot \nabla v = \int f v$$

$\eta \partial_i^2 u$ but this not $H_0^1(\Omega)$

SOLUTION:

$\Rightarrow$ $\delta_{h,k} u = \dfrac{u(x + h e_k) - u(x)}{h}$

Consider difference Quotient $\in H_0^1$

CACCIOPPOLI:

$$\Rightarrow \int \eta^2 |\nabla \delta_{h,k} u|^2 \leq \int |f|^2 + c \int |\delta_{h,k}|^2.$$

Take limits on both sides and apply CACCIOPPOLI again:

$$\int_{B_r(x_0)} |\partial^2 u|^2 \leq c \int_{B_R(x_0)} |f|^2 + c \int_{B_R(x_0)} |u|^2.$$

$\Rightarrow$ $u \in H^2_{loc}(\Omega)$.

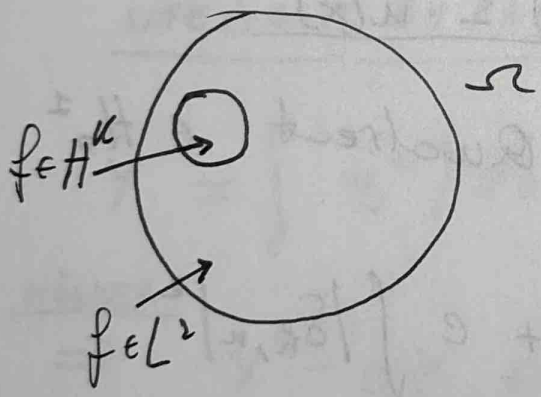
$\Rightarrow$

$$-\Delta u = f \in L^2 \quad \text{on } \Omega$$

$$u|_{\partial\Omega} = 0$$

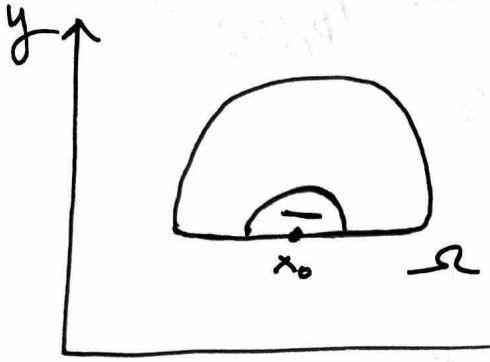
$$u \in H_0^1 \cap H^2_{loc}$$

- If $f \in H^k$, then $u \in H_0^1 \cap H^{k+2}_{loc}$
  (by moving all derivatries to f)
  & CACCIOPPOLI



- We don't use difference quotient instead of convolutions? ~~Because with convolution I have problems on the boundary~~ $\partial\Omega$.
- With Diff. quotients I have problem as I get closer to $\partial\Omega$!!

- **FLAT BOUNDARY:**
- $-\Delta u = f$, $f \in L^2$



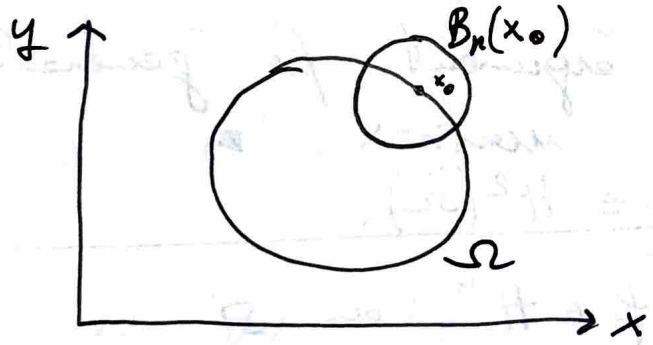
→ Can controll 2-derivatives in x & 1 in y + 1 mix By integration by parts!!

u_{xx}, u_{xy} CONTROLLED By u, f in L^2 (CACCIOPPOLI)

$$-\Delta u = -(u_{yy} + \Delta_x u) = f$$

⇒) Equation Tells me $u_{yy} \in H^2_{loc}$!!

- **NON-FLAT BOUNDARY:**



⇒ Take partition of unity, localize u to $B_R(x_0)$ and use change of variables to FLATTEN-OUT $\partial\Omega|_{B_R(x_0)}$

⇒) Use FLAT BOUNDARY ARGUMENT ABOVE!!

- To FLATTEN OUT $\partial\Omega|_{B_r(x_0)}$, use CHANGE of VARIABLES:

$$x \longrightarrow y$$

$$\frac{\partial u}{\partial x^i} = \frac{\partial u}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^i},$$

$$a_{\alpha,\beta} = \sum_i \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^i} \quad \text{(chain Rule)}$$

$$\Rightarrow \int a_{\alpha\beta}\frac{\partial y}{\partial y^\alpha}\frac{\partial u}{\partial y^\beta}$$

→ Use argument for general a_{ij} matrix. ■

$\Rightarrow u \in H^2(\Omega)$.

- **THM**: $-\Delta u = f \in H^k$ on Ω

$u|_{\partial\Omega} = 0$, $\partial\Omega \in C^{k+2}$, then

$$u \in H_0^1 \cap H^{k+2}.$$

- **THM**: $-\partial_i a^{ij} \partial_j u = f \in H^k$ on Ω,

$$C|\xi|^2 \geq a^{ij}\xi_i\xi_j \geq C|\xi|^2$$

$$u|_{\partial\Omega} = 0 \quad , \quad \partial\Omega \in C^{k+2}$$

then $u \in H_0^1 \cap H^{k+2}$

- **NEUMANN PROBLEM**:

- $-\Delta u = f \in L^2$ on Ω

$$n \cdot Du|_{\partial\Omega} = 0$$

$$\circledast \quad \int Du \cdot Dv = \int fv \quad , \quad \forall\, v \in \tilde{H}^1 . \qquad \left(\int_\Omega w = 0\right)$$

- We solved $\circledast$ by LAX MILGRAM.

- NOTICE: If we write $v = \eta^2(u - u_\Omega)$, nothing changes in technique used with cacciopoli above, since when differentiate u_Ω drops out!!

→ $v = \eta^2 (u - u_{av})$

→ Use same argument to get $v \in \tilde{H}^1 \cap H^2_{loc}$

→ Flattening of the boundary never used the boundary values: we same argument above to get $v \in \tilde{H}^1 \cap H^2$

→ Integrate by parts:

$$\int_{\partial\Omega} (n \cdot \nabla u) v - \int_\Omega \Delta u \, v = \int_\Omega f v$$

$$\int_{\partial\Omega} (n \cdot \nabla u) v = 0 \quad , \forall \, v \in \tilde{H}^1$$

$$\Downarrow$$

$$\boxed{n \cdot \nabla u = 0}$$

$\Rightarrow$ NOW CAN SOLVE NEUMANN PROBLEMS.

- **MIDTERM** :

(i): Lots of PDE's, can solve them? ∃! solution? (Cauchy-Kovalevsky, inequalities, Max-Principle for $\Delta u = 0$ ...)

(ii): Distributions, F.T.

(iii): Elliptic Theory, C^α regularity, learn Estimates (Singular Integral operators ESTIMATES).

- When is a problem elliptic? (Characteristics)
- What regularity theory we worked out?