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• **Class** (#1 after midterm)

$$D(u,v) = \int_\Omega \{ a_{ij}\, \partial_i u\, \partial_j v + b_i\, \partial_i u\, v + c_i\, \partial_i v\, u + d\, u\, v \}\, dx$$

→ For existence, need to apply L-M:

* $D(u,u) \geq c_1 \|u\|_{\underline{X}}^2$
* $|D(u,v)| \leq c_2 \|u\|_{\underline{X}} \|v\|_{\underline{X}}$
  - ↳ 2nd bound from above always satisfied if $\underline{X} \subseteq H^1(\Omega)$ (just use Cauchy-Schwartz/Hölder).

→ $\left\{ \begin{array}{l} D(u,v) = \langle f, v \rangle \quad , f \in L^2 \\ \text{Appropriate space } \underline{X} \text{ chosen} \end{array} \right\} \Rightarrow$ BOUNDARY CONDITIONS

→ Assuming $f \in L^2(\Omega)$, then can take $v \in \overline{X} \subset H^1(\Omega)$ and use ELLIPTIC REGULARITY (i.e., $u \in H^2(\Omega)$) to integrate by parts and bring back derivatives to u, i.e., from above D:

$$\int_\Omega \left(a_{ij} n^j \partial_i u + c_i n^i u \right) v \, dS = 0,$$

$\forall \; v \in \overline{X} \subseteq H^1(\Omega)$.

→ If $C_0^\infty \subset \overline{X}$, then

$$a_{ij} n^j \partial_i u + c_i n^i u = 0 \quad \text{A.E.}$$

and in the sense of $\underline{\underline{D'}}$!!

NOTE: If $f \in H^{-1}(\Omega)$ only, then the PDE $Lu = f$ is satisfied only in the $H^{-1}(\Omega)$-sense, and DO NOT HAVE SUFFICIENT REGULARITY to integrate by parts back to u.

- "Different Situation":

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ n \cdot \nabla u\big|_{\partial\Omega} = 0 \end{cases}$$

$\Rightarrow \underline{X} \subset H^1(\Omega)$.

$= \partial_x^4 + \partial_y^4 + 2\partial_x^2\partial_y^2 \cdots$

- EX: $\Delta^2 u = 0$, ELLIPTIC.

(. For Cauchy Problem need $u, \partial u, \partial^2 u, \partial^3 u\big|_{\partial\Omega}$).

(. For B.V.P., need $u\big|_{\partial\Omega}$).

. DIRICHLET PROBLEM: $\underline{X} \subset H^2$

. $u\big|_{\partial\Omega} \in H^{2-\frac{1}{2}}$ ✓

. $\partial u\big|_{\partial\Omega} \in H^{2-(1+\frac{1}{2})}$ ✓

. $\partial^2 u\big|_{\partial\Omega} \notin L^2$. PROBLEM!!

$\Rightarrow$

- $\Delta^2 u = f$
- $u|_{\partial\Omega} = 0$, $\boxed{X = H_0^2}$
- $n \cdot \nabla u|_{\partial\Omega} = 0$

Space of H^2 functions s.t. $u|_{\partial\Omega} = \partial u|_{\partial\Omega} = 0$.

$\longrightarrow$ Can solve this problem on $H_0^2 = X$.

- $\Delta^2 u = f$
- $u|_{\partial\Omega} = 0$ ← Can be incorporated in X
- ⊛ $\Delta u|_{\partial\Omega} = 0$ ← CANNOT be incorp. in the space

$$\Rightarrow X = \left\{ u \in H^2 \;\middle|\; u|_{\partial\Omega} = 0 \right\}.$$

⊛ will have to come out of the DIRICHLET FORM $B(u,v)$ AFTER integrating by parts !!

- In general, if I'm given ELLIPTIC EQN. of degree $(2 \cdot m)$, then space will always have to be $\boxed{H^m}$!! (Degree must always be EVEN for eqn. to be ELLIPTIC).

- $\sum_{k,\ell=1}^{m} a_{ij}\, \partial_i^k \partial_i^\ell u$

$\rightsquigarrow$ $\int \sum_{\substack{k=1 \\ i,j}}^{m} A_{ij}\, \partial_i^k u\, \partial_j^k v$

$\rightsquigarrow$ $\overline{X} \subset H^m$

$\rightsquigarrow$ B. Cond.'s with "$m-1$" derivatives or less describes $\overline{X}$.

$\rightsquigarrow$ $\sum_{k,\ell=1}^{m} a_{ij}\, \partial_i^k \partial_j^\ell u + \cdots = f$.

$\rightsquigarrow$ Back to previous EXAMPLE $\Rightarrow$ ...

• EX: $\Delta^2 u = f$ (*)

or $\partial_i^2 \partial_j^2 u = f$, v

? (ASK) ⇝ (assume for now $u, v \in C_0^\infty$ to integrate by parts to find $a(u,v)$).

⇝ $\int_\Omega \sum_{i,j} (\partial_i \partial_j u)(\partial_i \partial_j u) = a(u,v)$.

BUT BILINEAR FORM IS NON-UNIQUE!

⇝ $\int_\Omega \Delta u \, \Delta v = a(u,v)$ is also a BILINEAR FORM!!

→ WHICH ONE TO USE DEPENDS ON THE BOUNDARY CONDITIONS I NEED!!

I can also add & subtract a term to EQN (*) and integrate by parts only one of the terms to get DIFFERENT $a(u,v)$ again!!

→ BDRY CONDITIONS DETERMINE CORRECT BILINEAR FORM!!

- Once I choose $a(u,v)$ to determine EXISTENCE (via L-M), use ELLIPTIC REG. to integrate by parts & find an EQN. with BDRY CONDITIONS!!

- Consider
$$a(u,v) = \int_\Omega \Delta u \, \Delta v$$
$$= -\int_\Omega \partial_i \Delta u \, \partial_i v + \int_{\partial\Omega} (n^i \partial_i v) \Delta u$$
$$= \int_\Omega v(\Delta^2 u) - \int_{\partial\Omega} (n^i \partial_i \Delta u) v + \int_{\partial\Omega} (n^i \partial_i v) \Delta u$$

→ Can incorporate either
$$\partial_i v|_{\partial\Omega} = 0 \quad ? \quad (\star SV)$$
or $\Delta u|_{\partial\Omega} = 0$ (thanks to ELLIPTIC REGULARITY) ?

- EX: (Previous)

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ n \cdot \nabla u \big|_{\partial\Omega} = 0 \end{cases} \leftarrow \text{CANNOT BE INCORPORATED IN THE SPACE}$$

$\rightarrow \overline{X} \subset H^1(\Omega)$.

$\rightarrow D(u,v) = \int_\Omega \nabla u \cdot \nabla v$

$$= \int_\Omega -\Delta u\, v + \int_{\partial\Omega} (n \cdot \nabla u)\, v$$

✓

- EX:

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ n \cdot \nabla u + u \big|_{\partial\Omega} = 0 \end{cases}$$

$D(u,v)$ as above DOESN'T work for these BDRY COND.'s !!

$\Rightarrow$ Take $-\Delta u + a_i \partial_i u - a_i \partial_i u = f \Rightarrow$ ...

$$\rightsquigarrow B(u,v) = \int_\Omega \Big[\nabla u \cdot \nabla v + (a_i \partial_i u) v + (a_i \partial_i v) u + (\partial_i a_i) u v \Big]$$

This $B(u,v)$ works for our boundary conditions:

$$\rightarrow B(u,v) = \int_\Omega -\Delta u \, v + \int_{\partial\Omega} \Big(n \cdot \nabla u \, v \ldots + (a_i n^i) u v \Big)$$

$\Rightarrow$ Need to choose $a \cdot n = 1$ to get

$$n \cdot \nabla u + u \Big|_{\partial\Omega} = 0 \quad \checkmark$$

EX:

$$(*)\begin{cases} -\Delta u = f \\ n \cdot \nabla u|_{\partial\Omega} = 0 \end{cases}$$

$$D(u,v) = \int_\Omega \nabla u \cdot \nabla v$$

$$\underline{X} = H^1(\Omega)$$

$$D(u,u) \geq c\,|u|^2_{H^1} \quad (??)$$

Consider $\begin{cases} -\Delta u = 0 \\ n \cdot \nabla u|_{\partial\Omega} = 0 \end{cases} \Rightarrow$ 1 is a solution, easy!

→ Project away from 1 !!

→ Take w s.t. $\langle w, 1\rangle = 0$, i.e.,

$$\int_\Omega w\,dx = 0$$

$w \in H^1$, $\tilde{w} = w - \int_\Omega w \cdot \Rightarrow \underline{X} = \tilde{H}^1(\Omega)$

(??) → POINCARÉ TELLS ME $D(u,u) = \int \nabla u^2 \geq |u|^2_{\tilde{H}^1}$

To derive the equation,

take $\varphi \in C_0^\infty$ s.t. φ has mean zero $\Rightarrow \varphi \in \tilde{H}^1$, i.e., take

$$\varphi - \frac{1}{|\Omega|}\int_\Omega \varphi \quad \in \tilde{H}^1$$

$$\Rightarrow \int_\Omega \nabla u \cdot \nabla \varphi = \int f\,(\varphi - \langle \varphi \rangle_{av}) \qquad (\triangle)$$

$$\Rightarrow \int_\Omega -\Delta u\, \varphi = \int_\Omega f\,\varphi - \underbrace{\langle \varphi \rangle_{av} \int_\Omega f}_{\| \int \langle f \rangle_{av}\, \varphi\, dx}$$

$\left(+ \cdots \text{NEUMANN BDARY} = 0\right)$

$\Rightarrow$ If solved $\boxed{-\Delta u = f - \langle f \rangle_{av}}$

↳ This makes RHS $\perp$ to NULL space of Laplacian !!

FREDHOLM !!

$(\triangle)$ By L-M. $\exists$! solution to $(\triangle)$!!

// in $\tilde{H}^1(\Omega)$ // !!

$\Rightarrow$ $\begin{cases} -\Delta u = f \\ n \cdot Du|_{\partial\Omega} = 0 \end{cases}$

ON $\tilde{H}^1(\Omega)$

has a! SOLUTION

IFF $\langle f \rangle_\Omega = 0$.

If work on $H^1(\Omega)$ only $\Rightarrow$ LOSE UNIQUENESS

- **FREDHOLM ALTERNATIVE**:

Linear Algebra: $Ax = b$, $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

→ Need $R(A) = N(A^*)^\perp$ if A is not INVERTIBLE (i.e., $b \notin N(A^*)$).

- EXTEND FREDHOLM to general HILBERT SPACES:

→ Need A = Something Invertible + + Compact

= I − COMPACT

→ Need form of REGULARITY to get Compactness $\Rightarrow$ CAN ONLY APPLY THIS THEORY TO ELLIPTIC PROBLEMS!

→ HYPERBOLIC PROBLEMS HAVE NO GAIN OF REGULARITY !!
↝ CAN'T APPLY FREDHOLM THEORY.

- **FREDHOLM**:

T COMPACT OPERATOR: $H \longrightarrow H$,
H a Hilbert Space.

(i): $N(I-T)$ is FINITE DIM.

(ii): $\dim\left(N(I-T)\right) = \dim\left(N(I-T^*)\right)$.

(iii): $R(I-T) = N(I-T^*)^{\perp}$

⇓

$R(I-T)$ is a Hilbert space follows from (iii) (& (i)).

- $(I-T)u = f$ is SOLVABLE if (iii) is satisfied.