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- **Class**: (2nd after MIDTERM:)

- EIGENVALUE PROB. for LAPLACE'S EQ.
- HEAT EQN., SCHRÖDINGER EQN., WAVE EQN.

- $\overline{\Omega}$ COMPACT THROUGHOUT.
- $\mathcal{L} = -\Delta$ + BOUNDARY CONDITIONS ← OPERATOR.
- Consider $-\Delta$ with "$-$" to have a positive operator (w/ bdry conditions).
- Dirichlet, Neumann's, Robins.
- $-\Delta u = f$ + bdry conditions, the inverse is a compact operator.

Ex:

$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases} \Rightarrow u \in H_0^1 \cap H^2$$

$\Downarrow$

GAIN OF REGULARITY

$\Downarrow$

$(-\Delta)^{-1}$ is COMPACT!!

- Self-Adjoint operator A:

$$\langle Au, v \rangle = \langle u, Av \rangle$$

- Laplacian is <u>Self-adjoint</u> on $H_0^1 \cap H^2$:

$$\langle -\Delta u, v \rangle = -\int \Delta u\, v = -\int u\, \Delta v = \langle u, -\Delta v \rangle$$

$\Rightarrow$ $-\Delta$ has all real eigenvalues, etc..., all properties of self-adjoint operators.

- **A self-adjoint:**

→ Solve $\inf_{|u|^2=1} \frac{1}{2}\langle Au, u \rangle$

→ F.O.C. w/ Lagrange Multipliers:

$$A e_1 = \lambda_1 e_1$$

→ Solve $\inf_{\substack{|u|^2=1 \\ \langle u, e_1 \rangle = 0}} \frac{1}{2}\langle Au, u \rangle \Rightarrow Ae_2 = \lambda_2 e_2$, $\lambda_1 \leq \lambda_2$

→ Can do the same with $-\Delta$ since it's self-Adjoint!

→ $$\inf_{|u|_{L^2}^2 = 1} \frac{1}{2} \int |\nabla u|^2 \, dx =: m$$

let u_n be a minimizing sequence, which exists since integrand ≥ 0.

$\Rightarrow |u_n|_{H_0^1} \leq 1$ ($|\nabla u| \leq 1$ by assumption).

$\Rightarrow \exists$ subsequence $u_n \longrightarrow e_1$ in H_0^1

(By weak compactness)

$\Rightarrow$ Rellich Compactness gives

$u_n \xrightarrow{L^2} e_1 \in H_0^1$.

$\Rightarrow \frac{1}{2} \int |\nabla u_n|^2 \longrightarrow m$

$$\frac{1}{2} \int |\nabla e_1|^2 \overset{(*)}{\leq} \underline{\liminf} \frac{1}{2} \int |\nabla u_n|^2 = m$$

By lower semicontin. of weak limits

But $|e_1|_{L^2}^2 = 1 \Rightarrow (*)$ is $=$!! (otherwise proof $\Rightarrow\Leftarrow$!)

$\Rightarrow$ Have shown that have

WEAK CONV. (in H_0^1) + NORM CONV. (in L^2)

$\Downarrow$

Have STRONG CONV. (in H_0^1) & contrary!

$\Rightarrow$ Can find other eigenvectors as I did before for general symmetric operators by arguing in the minimization that

$$\langle e_1, u \rangle = 0 .$$

- $m > 0$ by neg. (+ POINCARÉ + H_0^1):

$$\frac{1}{2}\int |\nabla e_1|^2 \leq \underline{\lambda m} \cdot \frac{1}{2}\int |\nabla u_m|^2 = m$$

$\Rightarrow$ All eigenvalues are strictly positive

$\Rightarrow$ $\underline{-\Delta \text{ on } M \text{ a positive operator!!}}$

- This argument works for any SELF-ADJOINT ELLIPTIC OPERATORS

↳ Need ELLIPTICITY to have m bounded from below!!

$\Rightarrow$ Have shown that have

WEAK CONV. + NORM CONV.
(in H_0^1) (in L^2)

$\Downarrow$

Have STRONG CONV. A posteriori!
(in H_0^1) ■

$\Rightarrow$ Can find other eigenvectors as I did before for general symmetric operators by assuming in the minimization that

$$\langle e_1, u \rangle = 0 .$$

- $m > 0$ by ineq. (+ POINCARE + H_0^1):

$$\frac{1}{2}\int |\nabla e_1|^2 \leq \underline{\lim} \cdot \frac{1}{2}\int |\nabla u_n|^2 = m$$

$\Rightarrow$ All eigenvalues are strictly positive

$\Rightarrow$ $\underline{-\Delta \text{ is a positive operator!!}}$

- This argument works for any SELF-ADJOINT ELLIPTIC OPERATORS

↳ Need Ellipticity to have m bounded from below!!

- For Neumann Boundary Conditions:

$$\inf_{u \in H^1(\Omega)} \frac{1}{2} \int |\nabla u|^2$$

$$|u|_{L^2} = 1$$

→ solution $= 0$, since $u = \frac{1}{\sqrt{|\Omega|}}$ solves the problem.

→ the next eigenvector $\lambda_1 > 0$!

- For Evolution Problems:

$\partial_t u - Au = 0$, A a "matrix" (operator).

ex: $\partial_t u = \Delta u$, $x \in \Omega$ (or $x \in \mathbb{R}^n$)

$u|_{\partial\Omega} = 0$

ex: $\partial_t u = i \Delta u$

ex: $u_{tt} - \Delta u = 0$

- $\frac{\partial u}{\partial t} = Au$, A is SELF-ADJOINT MATRIX (OPERATOR)

$u(0) = u_0$

$\mathcal{L}$ = Laplace Transform

$u(t) = e^{At} u_0$

$s\mathcal{L}(u) - u_0 = A(\mathcal{L}u)$

$\Rightarrow (sI - A)\mathcal{L}(u) = u_0$

$\Rightarrow \mathcal{L}(u) = (sI - A)^{-1} u_0.$

$\Rightarrow u = \left(\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} (sI - A)^{-1} ds \right) u_0$

- $\frac{\partial u}{\partial t} = \Delta u$, $x \in \Omega$, Ω bdd.

  → Need boundary values to determine behaviour of "Δ" operator.

  → $u\big|_{\partial\Omega} = 0$.

→ $u(0) = u_0 = u(0,x)$.

- $-\Delta$ on $\mathbb{R}^n$ NOT COMPACT !!
  No eigenvalue decomposition !!
  $-\Delta$ on $\mathbb{R}^n$ NOT INVERTIBLE !!

EX: on $\mathbb{R}^1$: $-\partial_x^2$ on $H_0^1[0,1]$.

→ Eigenvectors are $e_n = \sin(n\pi x)$

→ $u \in H_0^1$, $u = \sum u_n e_n(x)$,

$u_n = \int_0^1 u(x)\, e_n(x)\, dx$

(e_n's are orthonormal basis for $H_0^1[0,1]$).

Ex: $-\partial_x^2$ on $\mathbb{R}$, $L^2(\mathbb{R})$

→ look for eigenvectors:

$$-\partial_x^2 u = \lambda u$$

→ $u_{xx} + \lambda u = 0$

→ $\lambda = 0 \Rightarrow u = a + bx$

$\lambda > 0 \Rightarrow u = A\cos(\sqrt{\lambda}x) + B\dfrac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}$

$\lambda < 0 \Rightarrow$ $\boxed{u = A\cosh(\sqrt{\lambda}x) + B\dfrac{\sinh(\sqrt{\lambda}x)}{\sqrt{\lambda}}}$

↓

Growing exponentially, not in $L^2 \Rightarrow \lambda < 0$ solutions are irrelevant.

$\Rightarrow$ Inverse of $-\partial_x^2$ is NOT BOUNDED !! $\Rightarrow$ ...

$\Rightarrow -\partial_x^2$ NOT COMPACT !!

... for $\lambda \geq 0$, can find approximate eigenvalues, but for $\lambda < 0$ solutions are UNBOUNDED !!

But if set was Ω COMPACT, it would be fine !!

(There are elements in spectrum which are not eigenvalues !!)

- $\begin{cases} \partial_t u = u_{xx} & \text{on } x \in [0,1] \\ u(0,t) = u(1,t) = 0, \quad u(x,0) = u_0(x). \end{cases}$

$\rightsquigarrow u \in H_0^1[0,1]$.

(Eigenvectors: $e_n(x) = \sqrt{2}\sin(n\pi x)$, $n = 1, 2, \dots$ — $\sqrt{2}$: NORMALIZATION FACTOR

$\Rightarrow \int_0^1 e_n^2 \, dx = 1$)

$\rightsquigarrow u(x,0) = u_0(x) = \sum_{n=1}^{\infty} a_n e_n(x)$.

Take $u_0(x) \in L^2[0,1]$, thus need $\sum a_n^2 < \infty$

$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} a_n e_n(x)}$, $t \geq 0$.

SEPARATION OF VARIABLES.

→ Solution as a <u>limiting procedure</u>.

With $u_0(x) \in L^2([0,1])$ only, can't satisfy boundary condition for $t=0$!! ⇒

(NB) i.e., if $u_0 \in L^2(\Omega)$ only, then $u(x,t) \in C((0,\infty), H_0^1)$ & not H_0^1 for $t=0$.
↪ For $t>0$: FINE!!

need $u_0(x) \in H_0^1([0,1])$ and $\sum_{n=1}^{\infty} n^2 a_n^2 < \infty$

$$u(x,t) = \sum a_n e^{-\pi^2 n^2 t} e_n(x),$$

- **Q:** In what sense $t \to 0 \Rightarrow u(\cdot,t) \to u_0$?? In L^2!!

- For $t \geq 0$, $u \in C([0,\infty), L^2) =$
  = "continuous in time with values in L^2!"
- For $t > 0$, $u \in C((0,\infty), H_0^1) =$
  = "continuous on $(0,\infty)$-time, with values in H_0^1!"

• check that for $t>0$, $u \in H_0^1[0,1]$:

→ Differentiate:

$$\sum_n \pi n e^{-\pi^2 n^2 t} a_n \cos(n\pi x),$$

$$\Uparrow$$

$\exp(n^2)$ dominates n

→ Differentiate again:

$$-\sum_n \pi^2 n^2 e^{-\pi^2 n^2 t} a_n \sin(n\pi x),$$

$$\Uparrow$$

$\exp(n^2)$ dominates n^2

$\vdots \Rightarrow$ $u(x,t) \in H_0^k(\Omega)$, $\forall k$ by PLANCHEREL!! ([redacted] x)

$\Rightarrow u \in C^\infty((0,\infty) \times \Omega)$.

$\Rightarrow$ $u(x,t) = e^{At} u_0$,

$S(t) = e^{At}$ = operator parametrized by t.

= Found by solving the PDE.

→ **PROPERTY:** • $S(t)$ defined for $t > 0$,

and • $S(t_1)\,S(t_2) = S(t_1 + t_2)$, $t_1, t_2 > 0$.

and • $S(0) = I$.

• **Q**: In what sense $S(t) \xrightarrow{t \downarrow 0} I$?? ($u_0 \in L^2$).

NORM: $\| S(t) - I \| \longrightarrow 0$ (← NOT TRUE!!)

⇓

STRONG: $| S(t)u_0 - u_0 |_{L^2} \longrightarrow 0$ ✓

⇓

WEAK: $\langle S(t)u_0, v \rangle \longrightarrow \langle u_0, v \rangle$, $\forall\ v \in L^2$. ✓

→ **C_0 - SEMIGROUP**

- $u_0 \in C^\infty$

- **Q**: $\frac{d}{dt} S(t) u_0 \overset{?}{=} \Delta S(t) u_0$

  → "$=$" holds iff $u_0 \in H_0^2 \cap H^2$.

  (→ Can solve any LINEAR DIFFUSION EQUATION !!)

- $\partial_t u = i \Delta u$. Do same thing,

  get eigenvalues $i\pi^2 n^2$

  ⇓

  $e^{i\pi^2 n^2 t}$

  ⇓

  Solution defined $\forall\, t \in \mathbb{R}$.

  ⇓

  C_0 - GROUP !

BUT: Lost all the regularity that I had before ⇒ no GAIN OF REG. FROM WHERE YOU START!!

$\Rightarrow$ BOUNDARY CONDITIONS WILL NEVER BE SATISFIED !!

(If $u_0 \in L^2 \Rightarrow$ cannot talk about boundary values).

- Relationship between positivity of FUNDAMENTAL SOLUTION & $\nexists$ of MAXIMUM PRINCIPLE
