• Class: (#3 after Midterm).

(Rauch Textbook).

• Evolution Eqn. on $\mathbb{R}^d$

→ On $\mathbb{R}^d$ easier because of $\mathcal{F}$:

• S - space, seminorms $N_p(u) := \sup_{\substack{x \in \mathbb{R}^d \\ |\alpha|+|\beta| \le p}} |x^\alpha D^\beta u|$

• S' = Space of tempered distributions, $\mathcal{F}$ on S' defined by Duality

• $H^s(\mathbb{R}^d) = \left\{ u \in S' : \int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty \right\}$

* For $s \ge 0$, H^s is space of functions.

* For $s < 0$, H^s is space of Tempered Distributions S' (reminder $S' \subset D'$ since S' behave more nicely at ∞).

- Analogy with ODE's:

$\dot{x} = F(x)$, $x \in \mathbb{R}^d$, F Lipschitz (locally)

$x(0) = x_0$ ●, $x^\circ(t)$

→ Use Picard's Iteration, so need convergence in some space:

→ $x^{n+1}(t) = \int_0^t F(x^n(s))\,ds + x_0$

→ Sequence $\{x^k(t)\}_k$, what space do I define for convergence?

*If x^n makes sense, then F must be integrable ⇒ x^n is continuous in-time (Cannot think of L^1 or L^2 since eqn. (ODE) must make sense POINTWISE!!)

→ Choose $C([0,T], \mathbb{R}^d)$

→ For convergence of PICARD'S ITERATION:

we chose space $C([0,T], \mathbb{R}^d)$

→ $\sup_{t\in[0,T]} \left| x^{n+1}(t) - x^n(t) \right| = d_n(T)$, $x \in \mathbb{R}^d$.

→ $\left| x^{n+1}(t) - x^n(t) \right| \leq \int_0^t \left| F(x^n(s)) - F(x^{n-1}(s)) \right| ds$

$(F$ Lipschitz$) \leq \int_0^t C \left| x^n(s) - x^{n-1}(s) \right| ds$

(If F only locally Lipschitz, take time steps small enough so that solution, which is continuous doesn't blow up!).

→ Take sup:

$d_n(T) \leq C T d_{n-1}(T)$, $CT < \frac{1}{4}$,

$\Rightarrow d_n(T) \leq \frac{1}{4} d_{n-1}(T) \leq \frac{1}{2^{2n}} \left| 2|x_0| \right|$

$\Rightarrow\Rightarrow$

$\Rightarrow$ ~~$|x_n(t)| \leq 2|x_0| \sum_{n=1}^{\infty} \frac{1}{4^n}$~~

$\Rightarrow \cdots$

$$|x_m(t) - x_0| \leq 2|x_0| \frac{1}{4} \cdot \frac{4}{3} = 2|x_0| + \sum_{1}^{\infty} \frac{1}{4^m}$$

$\Rightarrow \left(x_m \in B_2(x_0) \right).$

- For evolution eqn., space is chosen to be $\underline{C([0,T], H^s)}$.

  Must be a Banach space for general problems

- $|||u|||_s = \sup_{0 \le t \le T} |u|_{H^s}$

- If $u \in C([0,T], H^s)$, can define

RIEMANN INTEGRAL:

$$\int_a^b u(s)\, ds, \qquad a, b \in [0,T].$$

- Ex:

$$\frac{\partial u}{\partial t} = \Delta u \quad , \quad x \in \mathbb{R}^d$$

$$u(0) = u_0$$

$\rightsquigarrow$ $u \in C([0,T], H^s)$, decide a posteriori what T, s are.

- cannot do PICARD's iteration since

$$\frac{\partial u^{n+1}}{\partial t} = \Delta u^n$$

and $u \in H^s \Rightarrow \Delta u \in H^{s-2}$

$\Rightarrow$ lose regularity at each iteration!!

$\Rightarrow$ get out of space!!

$\Rightarrow$ Can look at $\mathcal{F}$ of PDE:

$$\mathcal{F}\left(\frac{\partial u}{\partial t}\right) = \mathcal{F}(\Delta u) = -|\xi|^2 \hat{u}$$

$$\| \ ?$$

$$\frac{\partial}{\partial t}\mathcal{F}(u)$$

- Since $u \in C([0,T], H^s)$, then

$$\frac{\partial u}{\partial t} = \Delta u \in C([0,T], H^{s-2})$$

$\Rightarrow$ $u \in C^1([0,T], H^{s-2})$

$\Rightarrow$ Can interchange derivative in t and $\int$ for $\mathcal{F}$

$\Rightarrow$ $$\frac{\partial \hat{u}}{\partial t} = -|\xi|^2 \hat{u}$$

$\Rightarrow$ $$\boxed{\hat{u}(\xi,t) = e^{-|\xi|^2 t}\,\hat{u}_0}$$

- $t<0$, then $\hat{u} \notin H^s$, for any $\hat{u}_0 \in H^s$
- $t>0$, $\hat{u}$ is well-defined for any $\hat{u}_0 \in H^s$ because of $e^{-|\xi|^2 t}$

$\Rightarrow$ For $t>0$, thanks to $e^{-|\xi|^2 t}$, $\hat{u}(\cdot,t) \in H^s$ for any $s>0$

$t>0$: $\Rightarrow \hat{u} \in C^\infty_x$

$\Rightarrow$ because of PDE, $u \in C^\infty_t$

$\rightsquigarrow$ Can solve PDE for any $T>0$

$\Rightarrow T=\infty$.

- $u(x,t) = \int e^{-|\xi|^2 t}\, \hat{u}_0(\xi)\, e^{ix\cdot\xi}\, d\xi$

↳ $u(x,t) = \int K(x-y,t)\, u_0(y)\, dy$,

$K(x,t) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/4t}$

($\hat{u} = \hat{f}(\xi)\hat{g}(\xi) \Rightarrow u = f * g$

$\Rightarrow$ Just compute $\int e^{-|\xi|^2 t} e^{i\xi\cdot x}\, d\xi$)

• SCHRODINGER EQN :

$$i\,\partial_t u - \Delta u = 0$$

$$i\,\partial_t \hat{u} + |\xi|^2 \hat{u} = 0$$

$$\partial_t \hat{u} - i|\xi|^2 \hat{u} = 0$$

$$\rightsquigarrow \quad \hat{u}(\xi, t) = e^{i|\xi|^2 t}\, \hat{u}_0(\xi)$$

→ If u_0 in H^s, then
u in H^s_x but not better than that !!

→ Choose space for u to be
$C(\mathbb{R}, H^s)$

$$u(x,t) = \frac{1}{(2\pi t)^{d/2}} \int e^{-\frac{i|x-y|^2}{4t}}\, u_0(y)\, dy,$$

$u \in C(\mathbb{R}, H^s)$

(ASK)

- **RMK:** For Heat & Schrödinger eq'n easy to show **UNIQUENESS** in space ($C(\mathbb{R}, H^s)$ for Schrödinger, $C([0,T], H^s)$ for Heat).

(Just take u_1, u_2 both ~~[illegible]~~. solution w/ same u_0 etc ----).

- **WAVE EQN:**

$$u_{tt} - \Delta u = 0$$

$$u(x,0) = u_0, \quad \partial_t u(x,0) = u_1$$

$$\rightarrow \hat{u}_{tt} + |\xi|^2 \hat{u} = 0,$$

$$\hat{u}(\xi,t) = \cos(|\xi|t)\,\hat{u}_0 + \frac{\sin(|\xi|t)}{|\xi|}\,\hat{u}_1$$

$$\cdot\ u_0 \in H^s \Rightarrow \left(\cos(|\xi|t)\,\hat{u}_0\right)^{\vee} \in H^s$$

(Ex: • $f \in H^s$, $\frac{1}{1+|\xi|} \hat{f} \in H^{s+1}$

• $\frac{1}{|\xi|} \hat{f} \notin H^{s+1}$)

For our solution,

$$\frac{\sin(|\xi| t)}{|\xi|} \longrightarrow t \quad \text{as} \quad |\xi| \to 0$$

$\Rightarrow$ Bound depend on t.

• $\Rightarrow$ If $\hat{u}_1 \in H^{s-1}$, then

$$\frac{\sin(|\xi| t)}{|\xi|} \hat{u}_1(\xi) \in H^s$$

$\Rightarrow$ Take $u_0 \in H^s_x$, $u_1 \in H^{s-1}_x$

(Ex: $\ddot{x} + a\dot{x} + cx = 0$

$\frac{x}{y} = \dot{x} \Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = -ay - cx \end{cases}$)

$\Rightarrow$

... For wave eqn., think of it as a system:

$$w = \begin{pmatrix} u \\ \frac{\partial u}{\partial t} \end{pmatrix} \in C(H^s) \times C(H^{s-1})$$

$$\Rightarrow \quad u \in C(H^s) \cap C^1(H^{s-1})$$

(A s u)

$$u_{tt} - \Delta u = 0, \qquad x \in \mathbb{R}^d$$

$$u(0) = u_0, \qquad \partial_t u = u_1$$

$$\underbrace{\left(\frac{\sin|\xi| t}{|\xi|} \hat{g}(\xi) \right)^\vee}_{!!\ R * g} = \begin{cases} \displaystyle\int_{x-t}^{x+t} g(s)\, ds, & x \in \mathbb{R}. \\ \displaystyle\frac{c}{\blacksquare} \int_{|x-y|<t} \frac{g(y)}{\sqrt{-|x-y|^2 + t^2}}\, dy, & x \in \mathbb{R}^2. \\ \displaystyle\frac{1}{4\pi t} \int_{|x-y|=t} g(y)\, dS(y), & x \in \mathbb{R}^3 \end{cases}$$

$$\left(\cos(|\xi| t)\, \hat{f}(\xi) \right)^\vee = \partial_t R * f$$

To sum up :

- $\frac{\partial u}{\partial t} - \Delta u = 0$, $\frac{1}{(2\pi t)^{d/2}} \int e^{-\frac{|x-y|^2}{4t}} u_0(y) dy$

  $\rightarrow$ $u_0 \in H^s$, $s \geq 0$, $e^{-|\xi|^2 t} \hat{u}_0$

- $i \partial_t u - \Delta u = 0$, $\frac{1}{(2\pi t)^{d/2}} \int e^{-i\frac{|x-y|}{4t}} u_0(y) dy$

  $\rightarrow$ $u_0 \in H^s$, $s \geq 0$, $e^{+i|\xi|^2 t} \hat{u}_0$

- $u_{tt} - \Delta u = 0$, $u = \partial_t R * u_0 + R * u_1$.

  $\rightarrow$ $u \in C(H^s) \cap C^1(H^{s-1})$, $s \geq 1$.

(ASK)

- $\frac{\partial u}{\partial t} = \Delta u + V(x)\, u$

- $u(0) = u_1$

→ Can't use $\mathcal{F}$ on product, It would give me convolutions!!

2 OPTIONS:

(1): $A = \Delta + V(x)$ → Study the spectrum.

→ Use Laplace Transform:

$$(s - \Delta - V)\,\tilde{u} = u_0\,,$$

$$\tilde{u} = \frac{1}{s - \Delta - V}\, u_0$$

(2): VARIATION OF PARAMETERS:

DUHAMEL'S FORMULA

EX ODE:

$$\begin{cases} \dot{x} = Ax + f(t, x(t)) \\ x(0) = x_0 \end{cases}$$

$$\Rightarrow x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} \underbrace{f(s, x(s))}_{f(s, x(s))}\, ds$$

... Use PICARD's ITERATION

$$x^{n+1} = Ax^n + f(t, x^n(t))$$

or

$$x^{n+1}(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} f(s, x^n(s))\, ds$$

↳ If e^{At} doesn't make me lose regularity, can use PICARD's for PDE.

- **Back to our PDE:**

- $\dfrac{\partial u^{n+1}}{\partial t} = \cancel{\Delta u^n} + V(x)\, u^n$ ⟵ lose regularity.

- $u^{n+1}(t) = e^{\Delta t} u_0 + \int_0^t e^{\Delta(t-s)} V u^n(s)\, ds$

(where $e^{\Delta t}$ = Fundamental solution to $\dfrac{\partial w}{\partial t} = \Delta w$,

i.e. $e^{\Delta t} = \left(e^{-|\xi|^2 t}\right)^\vee$)

$e^{\Delta t} : H^s \longrightarrow H^s$ is BOUNDED ⟹))

$$\Rightarrow \left| u^{n+1}(t) \right|_{H^s} \lesssim \left| u^{n}(t) \right|_{H^s} .$$

- $\boxed{\text{CONTINUATION ARGUMENT}}$

$\rightsquigarrow$ Suppose T is FINITE : Consider

$$u(t) = e^{\Delta(t-T)} u(T) + \int_T^t e^{\Delta(t-s)} v \hat{u}(s)\, ds$$

$\Rightarrow \exists \; t > T$ s.t. solution exists $\Rightarrow T = \infty$.