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- **Class**: (# 4 after midterm)

MY SYNTHESIS w/ (USEFUL) MISTAKES → Fixed those in Class #5

→ **EVOLUTION EQUATIONS:**

(1): LOCAL EXISTENCE: PICARD's ITERATION (Contraction Mapping Principle).

(2): GLOBAL EXISTENCE: Energy estimates/ A priori estimates.

Ex: $\frac{\partial u}{\partial t} = \Delta u + |\nabla u|^2$, $x \in \mathbb{R}^3$

→ Need to choose a space $\overline{X}$ s.t. $|\nabla u|$ forms an Algebra under norm $|\cdot|_{\overline{X}}$, i.e.,

$$\left| |\nabla u|^p \right|_{\overline{X}} \lesssim \left| |\nabla u| \right|_{\overline{X}}^p \quad (*)$$

(If this holds for $p=2$, then go to any p by INDUCTION (HW #8)).

→ If can choose such $(\overline{X}, |\cdot|_{\overline{X}})$ so that ⊛ holds, then PICARD'S ITERATION WORKS!!

→ WRONG ITERATION:

$$\frac{\partial u^{k+1}}{\partial t} = \Delta u^k + |Du^k|^2 \quad \otimes$$

$u^k \in H^s \Rightarrow \Delta u^k \in H^{s-2}$

$\Rightarrow$ LOSE REGULARITY AT EACH ITERATION

(2 derivatives!! $\Rightarrow$ CAN DO BETTER!!).

→ CORRECT ITERATION:

$$\frac{\partial u^{k+1}}{\partial t} = \Delta u^{k+1} + |Du^k|^2.$$

NEED TO GAIN only 1 derivative! If can have such a gain, then $\overline{X} \longrightarrow \overline{X}$

$\Rightarrow$ PICARD'S WORKS!!

→ Goal now is to show LOCAL EXISTENCE of solution.

→ $\frac{\partial V}{\partial t} = \Delta V + F$, linear problem,

USE DUHAMEL'S PRINCIPLE:

$$V(t) = e^{\Delta t} V_0 + \int_0^t e^{\Delta(t-s)} F(s)\, ds$$

↳ Need to gain 1 DERIVATIVE since $F(s) = \left(|DV(s)|^{\bullet} \right)^2$ so that ITERATION MAPS $\overline{X} : \longrightarrow \overline{X}$.

→ Use ESTIMATE $\left| e^{\Delta t} \right|_{H^{\ell} \to H^{\ell+1}} \lesssim \frac{1}{\sqrt{t}}$

(showed in HW #8, for $\ell \geq 0$).

$$|V(t)|_{H^{\ell+1}} \lesssim |V_0|_{H^{\ell+1}} + \int_0^t \left| e^{\Delta(t-s)} F(s) \right|_{H^{\ell+1}} ds$$

$$\lesssim |V_0|_{H^{\ell+1}} + \int_0^t \frac{1}{(t-s)^{1/2}} |F(s)|_{H^{\ell}} ds$$

$\Longrightarrow \cdots$

... Notice that

$$|F(s)|_{H^\ell} = \left|\left(|\nabla v(s)|\right)^2\right|_{H^\ell}$$

$$\lesssim |\nabla v(s)|^2_{H^\ell}$$

for $\ell > \frac{d}{p}$, in this case $p=2$, $d=3$

$\Rightarrow$ $\ell > \frac{3}{2}$ guarantees H^ℓ is an algebra !! (HW #8).

$$\Rightarrow \quad |F(s)|_{H^\ell} \lesssim |\nabla v(s)|^2_{H^\ell}$$

$$\lesssim |v(s)|^2_{H^{\ell+1}}$$

- Back to our iteration:

For $\ell = 2 \Rightarrow H^{\ell+1} = H^3$:

$$|v(t)|_{H^3} \lesssim |v_0|_{H^3} + \int_0^t \frac{1}{(t-s)^{1/2}} |v(s)|^2_{H^3}\, ds$$

$$\lesssim |v_0|_{H^3} + \sqrt{t}\, \underbrace{|||v|||^2_{H^3}}_{\left(\sup_{s\in[0,t]} |v(s)|_{H^3}\right)^2}$$

LOC. LIP. FUNCTION of H^3-NORM ✓

$\Rightarrow$ PICARD'S ITERATION WORKS on $H^3(\mathbb{R}^3)$ by LAST BOUNDS FOUND!! DETAILS BELOW! ...

...$\rightarrow$ Define

$$\phi(v)(t) := e^{\Delta t} v_0 + \int_0^t e^{(t-s)\Delta} F(s)\, ds.$$

- $$\left| e^{\Delta t} v_0 - e^{\Delta t_0} v_0 \right|_{H^s} =$$

$$= \left| \left(e^{\Delta(t-t_0)} - I \right) \underbrace{e^{\Delta t_0} v_0}_{\in H^s} \right|_{H^s} \xrightarrow{t \downarrow t_0} 0$$

by STRONG CONTINUITY of $e^{\Delta t}$ on H^s which can be easily shown:

$$\left| e^{\Delta t} v_0 - v_0 \right|_{H^s} = \int (1+|\xi|^2)^s \left[e^{-2|\xi|^2 t} - 1 \right] \hat{v}_0^2\, d\xi$$

$$\leq |v_0|_{H^s} \sup_{|\xi| \in \mathbb{R}} \left\{ e^{-2t|\xi|^2} - 1 \right\}$$

$$\longrightarrow 0, \quad \text{as } t \to 0.$$

- $$\left| \int_0^t e^{(t-s)\Delta} F(s)\, ds - \int_0^{t_0} e^{(t_0-s)\Delta} F(s)\, ds \right|_{H^3} =$$

$$= \left| \int_{t_0}^t e^{(t-s)\Delta} F(s)\, ds + \int_0^{t_0} \left(e^{(t-t_0)\Delta} - I \right) e^{(t_0-s)\Delta} F(s)\, ds \right|_{H^3}$$

$\leq$ ...

$$\cdots \leq \int_{t_0}^{t} \left| e^{(t-s)\Delta} F(s) \right|_{H^3} ds +$$

$$+ \int_{0}^{t_0} \left| \left(e^{(t-t_0)\Delta} - I\right) \underbrace{e^{(t_0-s)\Delta} F(s)}_{\in H^3} \right|_{H^3} ds$$

$$\lesssim \int_{t_0}^{t} \underbrace{\left| F(s) \right|_{H^3}}_{\wedge \infty} ds +$$

$$+ \int_{0}^{t_0} \underbrace{\left| \left(e^{(t-t_0)\Delta} - I\right) e^{(t_0-s)\Delta} F(s) \right|_{H^3}}_{\in H^3} ds$$

$$\lesssim \left| F(s) \right|_{H^3}$$

$\Rightarrow$ If $v \in C([0,T], H^3(\mathbb{R}^d))$, by discussion above, with L.D.C.T., gives:

$$\left| \int_0^t e^{\Delta(t-s)} F(s)\, ds - \int_0^{t_0} e^{\Delta(t_0-s)} F(s)\, ds \right|_{H^3}$$

$$\xrightarrow{t \to t_0^+} 0$$

$\Rightarrow$ ...

∞ Therefore, on $\mathbb{R}^3$:

$$\phi : C([0,T], H^3) \longrightarrow C([0,T], H^3).$$

$$\rightarrow \phi(v_0)(t) = e^{\Delta t} v_0 + \int_0^t e^{(t-s)\Delta} |\nabla v_0|^2 \, ds$$

$$\rightarrow |\phi(v_0)(t) - v_0|_{H^3} \leq |e^{\Delta t} v_0 - v_0|_{H^3} +$$

$$+ \int_0^t |e^{(t-s)\Delta} |\nabla v_0|^2|_{H^3} \, ds$$

$$\leq \| e^{\Delta t} v_0 - v_0 \|_{H^3} +$$

$$+ \sqrt{T} \, \| v_0 \|_{H^3} = K.$$

$\rightarrow$ Choose T s.t. $K = 1$..

$\rightarrow$ Let $v^{n+1}(t) = \phi(v^n)(t)$. Want to show

(i): $\| v^{n+2} - v^n \|_{H^3} < q \, \| v^n - v^{n-1} \|_{H^3}$, $q \in [0,1)$ and

(ii): $\{v^n\}_n \subset \overline{X}$, $\overline{X}$ a CLOSED SUBSPACE of $C([0,T], H^3)$ so that can apply BANACH-THM.

(Since $F(\cdot) = (\cdot)^2$, can use LOCALLY LIPSCHITZ to get a contraction, same as for ODE's)
Therefore,

$$\left| v^{n+2}(t) - v^{n}(t) \right|_{H^3} \leq \int_0^t \left| e^{(t-s)\Delta} \left| F(|\nabla v^n(s)|) - F(|\nabla v^{n-2}(s)|) \right) \right|_{H^3} ds$$

$$\lesssim \int_0^t \frac{1}{(t-s)^{3/2}} \left| F(|\nabla v^n(s)|) - F(|\nabla v^{n-2}(s)|) \right|_{H^2} ds$$

$$\leq L \int_0^t \frac{1}{(t-s)^{1/2}} \left| |\nabla v^n(s)| - |\nabla v^{n-1}(s)| \right|_{H^2} ds$$

$$\lesssim L \int_0^T \frac{1}{(T-s)^{1/2}} \left| |v^n(s)| - |v^{n-1}(s)| \right|_{H^3} ds$$

$$\leq L\sqrt{T} \left\| v^n - v^{n-1} \right\|_{H^3}$$

$$\Rightarrow \left\| v^{n+1} - v^n \right\|_{H^3} \leq L\sqrt{T} \left\| v^n - v^{n-1} \right\|_{H^3},$$

where $L\sqrt{T} < \frac{1}{2}$ for an appropriate choice of T. Then, $\{v^n\}_n \subset \overline{B_2(v_0)}^{H^3}$, L is Lipschitz constant for such space. (Same as for ODE) $\Rightarrow$ ...

∴ Then, by Banach Fixed Point Theorem,

$\exists!$ $u(t) \in C([0,T], \mathbb{H}^3)$ so that PDE is SATISFIED on the space.

(NOTATION : $|||v|||_{HS} = \sup_{t\in[0,T]} |v(t)|_{HS}$).

→ LOCAL EXISTENCE ✓

• In ODE's, for GLOBAL EXISTENCE need either 1 – GLOBALLY LIPSCHITZ $F(\cdot)$

or 2 – UNIFORM BOUND for solution in time.

EX: $\ddot{x} + x + \lambda x^3 = 0$

$$\Rightarrow \frac{d}{dt}\underbrace{\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\lambda}{4}x^4\right)}_{\text{→ CONSERVED Q.TY.}} = 0$$

$$\Rightarrow \frac{1}{2}|\dot{x}(t)|^2 + \frac{1}{2}|x(t)|^2 + \frac{\lambda}{4}|x(t)|^4 = \frac{1}{2}|\dot{x}(0)|^2 + \frac{1}{2}|x(0)|^2 + \frac{\lambda}{4}|x(0)|^4.$$

$\Rightarrow$ Have $|x(t)| \leq C(x(0), \dot{x}(0)) \quad \forall\, t \geq 0.$

$\Rightarrow$ GLOBAL BOUND on SOLUTION!

- **CONTINUATION ARGUMENT for ODE's:**

$$\dot{x} = Ax + f(x), \qquad |x(t)| \leq C(x(0))$$
$$x(t_0) = x_{t_0}.$$

$\rightarrow$ $$\phi(v)(t) = e^{A(t-t_0)} x_{t_0} + \int_{t_0}^{t} e^{A(t-s)} f(v(s))\, ds.$$

$\rightarrow$ Use Picard's iteration $x^{n+1}(t) = \phi(x^n)(t)$.

- By UNIFORM BOUND on $|x(t)|$, can choose T^1 so small so that $|x^1 - x_{t_0}| < 2$ and $|x^n - x_{t_0}| < 2$, $\forall\, n \geq 1$ and s.t. ϕ is a contraction on $C([t_0, t_0 + T^1], B_2(x_{t_0}))$.
- Choice of T^1 is INDEPENDENT of t_0 (where we are starting from in time)

$\Rightarrow$ can Repeat this process until $T_{max} = \infty \Rightarrow$ GLOB. EXISTENCE!

■

→ For PDE's, if have an a PRIORI BOUND on $|u|_{H^3}$ (in our last PDE EXAMPLE), then CONTINUATION ARGUMENT APPLIES !!

→ GLOBAL EXISTENCE !!

Ex: $\frac{\partial u}{\partial t} = \Delta u + \lambda u^3$, $x \in \mathbb{R}^3$,

$$\frac{\partial u^{k+1}}{\partial t} = \Delta u^{k+1} + \lambda (u^k)^3$$

→ From #W#8, choose $\ell = 2 > \frac{3}{2}$ so that $H^2(\mathbb{R}^3)$ is an algebra

⇒ Convergence of PICARD'S ITERATION on H^2

⇒ LOCAL EXISTENCE !

→ For GLOBAL EX., need A PRIORI BOUND on $|u|_{H^2}$, i.e., independent of time.

- **A, B, C - Method for A PRIORI BOUND:**

$$A u_t + B \cdot \nabla u + C u$$

(i): Multiply PDE by u: (Always assume $u(x) \to 0$ as $|x| \to \infty$, $u \in L^2$ ✓)

$$\to \partial_t \left(\frac{u^2}{2} \right) = u(\partial_i^2 u) + \lambda u^4$$

$$= \partial_i(\partial_i u \, u) - |\nabla u|^2 + \lambda u^4$$

→ Finally let Ω_n be a ball at zero of radius n:

$$\partial_t \int_{\Omega_n} \frac{1}{2} u^2 = \int_{\Omega_n} \partial_i (\partial_i u \, u) - \int_{\Omega_n} (|\nabla u|^2 + \lambda |u|^4)$$

$$= \int_{\partial \Omega_n} \vec{n}^i u \, \partial_i u - \int_{\Omega_n} (|\nabla u|^2 + \lambda |u|^4)$$

Integrate in t: $\Longrightarrow$ ...

... let $n \to \infty$:

$$\int \frac{1}{2} u(t)^2 \, dx + \int_0^t \int \left(|\nabla u(s)|^2 - \lambda |u(s)|^4 \right) dx \, ds = \int \frac{1}{2} u(0)^2 \, dx \,, \quad \forall\, t \geq 0$$

$\Rightarrow$ For $\lambda = -1$, we have :

- $|u(t)|_{H^1} \lesssim |u(0)|_{H^1}$
- $|\nabla u|_{L_t^2 L_x^2} \lesssim |u(0)|_{H^1}$
- $|u|_{L_t^4 L_x^4} \lesssim |u(0)|_{H^1}$.

(ii): Multiply by u_t :

$$u_t^2 = u_t \partial_i^2 u + \lambda u_t u^3$$

$\Leftrightarrow$ $$u_t^2 = \partial_i (u_t \partial_i u) - \frac{1}{2} \partial_t |\nabla u|^2 + \frac{\lambda}{4} \partial_t (u^4)$$

$\Rightarrow$ ...

∞ By same procedure as above, among $|u(x)| \to 0$ as $|x| \to \infty$:

$$\int |u_t|^2 + \partial_t \int \frac{1}{2} |\nabla u|^2 - \frac{\lambda}{4} \partial_t \int |u|^4 =$$

$$= 0$$

Integrate in t:

$\Rightarrow$

$$\int_0^t \int |u_t(s)|^2 \, dx \, ds + \int \frac{1}{2} |\nabla u(t)|^2 \, dx - \frac{\lambda}{4} \int |u(t)|^4 \, dx =$$

$$= \int \frac{1}{2} |\nabla u(0)|^2 \, dx + \frac{\lambda}{4} \int |u(0)|^4 \, dx$$

→ By Sobolev Interp. Ineq, let $j = 0$, $p = q = 2 \Rightarrow \alpha = \frac{3}{4} \Rightarrow$ can bound

$$|u|_{L^4} \leq |Du|_{L^2}^{\alpha} |u|_{L^2}^{1-\alpha}$$

→ $u(0) \in H^2 \subset H^1 \Rightarrow u(0) \in L^4$.

$\Rightarrow \cdots$ **Thus, for $\lambda = -1$:**

$$\| u_t \|_{L_t^2 L_x^2} \lesssim \| u(0) \|_{H^1} .$$

$\Rightarrow$ **To sum up: ($\lambda = -1$)**

A PRIORI BOUNDS

- $\| u(t) \|_{H^1} \lesssim \| u(0) \|_{H^1}$
- $\| \nabla u \|_{L_t^2 L_x^2} \lesssim \| u(0) \|_{H^1}$
- $\| u_t \|_{L_t^2 L_x^2} \lesssim \| u_0 \|_{H^1}$

(and $\| u \|_{L_t^4 L_x^4} \lesssim \| u(0) \|_{H^1}$),

where $\boxed{u_t = \Delta u - u^3 \quad , \quad x \in \mathbb{R}^3}$

→ We established local existence by doing PICARD ITERATION on $H^2(\mathbb{R}^3)$

Our estimates are not good enough to show GLOBAL EXISTENCE, we would need A PRIORI BOUND $|u(t)|_{H^2} \leq |u_0|$ but we only have bounds on $H^1(\mathbb{R}^3)$!

IDEA :

GO BEYOND ODE THEORY, try to show GLOBAL EXISTENCE on $H^1(\mathbb{R}^3)$!!

$$u(t) := e^{\Delta t} u_0 - \int_0^t e^{\Delta(t-s)} u^3(s)\, ds$$

→ Do Picard's ITERATION on $H^1(\mathbb{R}^3)$

(TRY & see!)

... CONTRACTION ARGUMENT on $H^1(\mathbb{R}^3)$:

$$|u(t)|_{H^1} \lesssim |u_0|_{H^1} + \int_0^t \left| e^{\Delta(t-s)} u^3(s) \right|_{H^1} ds$$

→ look at H^1 derivative only, since can always bound L^2 terms (Sobolev Ineq. ...). Thus,

$$|\nabla u(t)|_{L^2} \lesssim |\nabla u_0|_{L^2} + \int_0^t \underbrace{\left| \nabla e^{\Delta(t-s)} u^3(s) \right|_{L^2}} ds$$

$$\left| e^{\Delta(t-s)} \nabla(u^3(s)) \right|_{L^2} \lesssim |\nabla(u^3(s))|_{L^2}$$

→ $|\nabla(u^3(s))|_{L^2}$, cannot bound this since $\ell = 1 < \frac{3}{2}$, i.e., $H^1(\mathbb{R}^3)$ NOT AN ALGEBRA.

→ $|\nabla u(s)\, u^2(s)|_{L^2}^2$, cannot bound this either, since ℓ-S. gives me $\lesssim |u^2(s)|_{L^8} |\nabla u(s)|_{L^4}$

⇒ ...

... Thus, MUST SACRIFICE u SINGULARITY in t to get BETTER estimates in x:



$$\left| e^{\Delta(t-s)} u^3(s) \right|_{H^1} \lesssim \frac{1}{(t-s)^{1/2}} \left| u^3(s) \right|_{L^2}$$

$$= \frac{1}{(t-s)^{1/2}} \left| u(s) \right|_{L^6}^3$$

$$\left(\begin{matrix}\text{SOBOLEV} \\ \text{EMB.}\end{matrix}\right) \longrightarrow \lesssim \frac{1}{(t-s)^{1/2}} \left| Du(s) \right|_{L^2}^3$$

$$\lesssim \frac{1}{(t-s)^{1/2}} \left| u(s) \right|_{H^1}^3 .$$

Thus, conclude that

$$\left| u(t) \right|_{H^1} \lesssim \left| u_0 \right|_{H^1} + \int_0^t \frac{1}{(t-s)^{1/2}} \left| u(s) \right|_{H^1}^3 ds$$

$$\lesssim \left| u_0 \right|_{H^1} + \sqrt{t} \left| \left| \left| u \right| \right| \right|_{H^1}^3$$

$\Rightarrow$ GET LOCAL EXISTENCE BY PICARD ITERATION on $C([0,T], H^1(\mathbb{R}^3))$ and BY PREVIOUS (UNIFORM) APRIORI ESTIMATE GET GLOBAL EXISTENCE $\Rightarrow T_{max} = \infty$ ■

... Thus, MUST SACRIFICE SINGULAR in t " to get BETTER estimates in x :

$$\left| e^{\Delta(t-s)} u^3(s) \right|_{H^1} \lesssim \frac{1}{(t-s)^{1/2}} \left| u^3(s) \right|_{L^2}$$

$$= \frac{1}{(t-s)^{1/2}} \left| u(s) \right|_{L^6}^3$$

$$\underbrace{\text{(SOBOLEV. EMB.)} \rightarrow}_{} \lesssim \frac{1}{(t-s)^{1/2}} \left| Du(s) \right|_{L^2}^3$$

$$\lesssim \frac{1}{(t-s)^{1/2}} \left| u(s) \right|_{H^1}^3 .$$

Thus, conclude that

$$\left| u(t) \right|_{H^1} \lesssim \left| u_0 \right|_{H^1} + \int_0^t \frac{1}{(t-s)^{1/2}} \left| u(s) \right|_{H^1}^3 ds$$

$$\lesssim \left| u_0 \right|_{H^1} + \sqrt{t} \, ||| u |||_{H^1}^3$$

$\Rightarrow$ GET LOCAL EXISTENCE BY PICARD ITERATION on $C([0,T], H^1(\mathbb{R}^3))$ and BY PREVIOUS (UNIFORM) APRIORI ESTIMATE GET GLOBAL EXISTENCE $\Rightarrow T_{max} = \infty$ ■

$\Rightarrow$ Thus, have GLOBAL SOLUTION on #1.

• GRONWALL'S INEQUALITY:

Let $x(t)$ be NONNEGATIVE, ABSOLUTELY CONTINUOUS on $[0,T]$ s.t. for A.E. t it satisfies:

$$\dot{x}(t) \le b(t)\,x(t), \qquad b(t) > 0, \text{ integrable on } [0,T].$$

(N.B.)

Then,

$$x(t) \le e^{\int_0^t b(s)\,ds}\, x(0), \quad \forall\, t \in [0,T].$$

PROOF:

$$\frac{d}{dh}\left(e^{-\int_0^h b(s)\,ds}\, x(h) \right) =$$

$$= e^{-\int_0^h b(s)\,ds} \left(\dot{x}(h) - b(h)\,x(h) \right)$$

$$\le e^{-\int_0^h b(s)\,ds} \cdot 0 = 0$$

$$\Rightarrow e^{-\int_0^t b(s)\,ds}\, x(t) \le x(0)$$

$$\Rightarrow x(t) \le x(0)\, e^{\int_0^t b(s)\,ds} \quad \blacksquare, \ \forall\, t \in [0,T]$$

- Suppose $u_0 \in H^k(\mathbb{R}^3)$, $k \geq 1$, Can I get $H^k(\mathbb{R}^3)$ – BOUNDS for $u(t)$?

EX: $k=2$. Then, by discussion above

$$|u(t)|_{H^2} \leq |u_0|_{H^2} + \int_0^t \frac{1}{\sqrt{(t-s)}} |u^3(s)|_{H^1} ds,$$

(where ... : $|u^3|_{L^2} = |u|_{L^6}^3 \lesssim |Du|_{L^2}^3$,

$|Du^3|_{L^2} = |u^2 Du|_{L^2} \lesssim |u^2|_{L^\infty} |Du|_{L^2}^2$

$\lesssim |u|_{H^2}^2$)

$$\rightsquigarrow |u(t)|_{H^2} \lesssim |u_0|_{H^2} + \int_0^t \frac{1}{\sqrt{(t-s)}} |u(s)|_{H^2}^2 ds$$

$\rightsquigarrow$ Use GRONWALL INEQ. to obtain

$$|u(t)|_{H^2} \lesssim e^{\sqrt{t}} |u_0|_{H^2}$$

$\rightsquigarrow$ For any fixed $T \geq 0$, $u \in C([0,T], H^2)$, i.e., $\boxed{u(t) \in H^2(\mathbb{R}^3), \forall\ t \geq 0.}$!!

→ For any $k \geq 2$, on $\mathbb{R}^3$, $H^k(\mathbb{R}^3)$ forms an algebra

$\Rightarrow$ $\boxed{u_0 \in H^k(\mathbb{R}^3) \Rightarrow u(t) \in H^k(\mathbb{R}^3), \quad \forall\, t \geq 0.}$

⊛ TO SUM UP:

(i): We got existence GLOBALLY only on $H^1(\mathbb{R}^3)$ since we could rely only on H^1 - A PRIORI ESTIMATE

(ii): Once GLOBAL EXISTENCE has been established, we verified that $u(t) \in H^k(\mathbb{R}^3)$, $\forall\, t \geq 0$ thanks to Gronwall Ineq.