• **Class**: (# 4 after midterm)

Evolution Equations

(1): Local Existence: Picard Iteration (Contraction Mapping Principle)

(2): Global existence: Energy estimates / A priori Estimates

Ex: $\frac{\partial u}{\partial t} = \Delta u + |Du|^2$, $x \in \mathbb{R}^3$

• 1st choose a space s.t. $|Du|_{\underline{X}}$ is an algebra:

$$|(Du)^p|_{\underline{X}} \leq |Du|^p_{\underline{X}}$$

(do it for $p=2$ $\Rightarrow$ any p by induction)

$\Rightarrow$ If can choose such norm/space then PICARD'S ITERATION WORKS.

(↙ if just only u, then lose 2-derivatives at each iteration)

$$\rightarrow \frac{\partial u^{k+1}}{\partial t} = \Delta u^{k+1} + |\nabla u^k|^2$$

$$\rightarrow \frac{\partial v}{\partial t} = \Delta v + F \quad , \text{ linear problem,}$$

use <u>Duhamel's Principle:</u>

$$v(t) = e^{\Delta t} v_0 + \int_0^t e^{\Delta(t-s)} F(s)\, ds$$

→ Estimate v in terms of F.

→ <u>Need to gain one derivative</u>

in order to iterate since $F = |\nabla \cdot|^2$

→ Use estimate (3)(d) from Hw #8:

$$|v|_{H^{\ell+1}} \leq |v_0|_{H^{\ell+1}} + \int_0^t \frac{C}{(t-s)^{1/2}} |F|_{H^\ell}\, ds,$$

$$\ell > \frac{d=3}{2} \Rightarrow \underline{H^{\ell+1} \text{ is an ALGEBRA}} \text{ By problem (\#1) in Hw \#8.}$$

$\Rightarrow$ $\ell = 2$ is fine

$\Rightarrow$ Iterate on $H^{\ell+1}$ $\Rightarrow$ (N.B.) $u^{k+1}(0) = u_0 \in H^3$

→ We got LOCAL EXISTENCE, but no GLOBAL EXISTENCE !! (YET)

→ In ODE's, for GLOBAL EXISTENCE need either "F" GLOB. LIPSCHITZ or Lyapunov functions.

EX: $\ddot{x} + \dot{x} + \lambda x^3 = 0$

→ $\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\lambda}{4}x^4\right) = 0$

→ $\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\lambda}{4}x^4 = \frac{1}{2}|\dot{x}(0)|^2 + \frac{1}{2}|x(0)|^2 + \frac{\lambda}{4}|x(0)|^4$

→ have a BOUND independent of time !! ⇒ CONTINUATION ARGUMENT WORKS!

⇒ GLOB. EXISTENCE!

- CONTINUATION ARGUMENT for ODE :

$$\dot{x} = Ax + f(x), \qquad \boxed{|x(t)| \le C(x_0)} \qquad x_0 = x(0).$$

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-s)} f(x(s))\, ds$$

~~$|x(t)| \le C x_0 + \int_{t_0}^{t} e^{A(t-s)}$~~

$$\rightarrow |x(t)| \le C(x_0) + \int_{t_0}^{t} \tilde{C}$$

$[t_0, t_0+T]$, $C(x_0)\,T < 1$, Need

$$T < \frac{1}{C(x_0)}$$

for PICARD's Convergence.

$\Rightarrow$ For our previous PDE, if have A PRIORI BOUND on $|v|_{H^3}$ $\rightarrow$ Continuation Argument applies!! $\rightarrow$ GLOBAL.

Ex: $\dfrac{\partial u}{\partial t} = \Delta u + \lambda u^3, \quad x \in \mathbb{R}^3$

→ $\dfrac{\partial u}{\partial t}^{k+1} = \Delta u^{k+1} + \lambda \left(u^k\right)^3,$

Need iteration on H^2,

$2 > \frac{3}{2}$ ✓

→ H^2 is an ALGEBRA

→ Convergence of PICARD's ITERATION & get LOCAL EXISTENCE

→ **BUT**: for GLOBAL EX. **need**

A PRIORI BOUND on $|u|_{H^2}$.

(independent of time).

• Look at $\boxed{\frac{\partial u}{\partial t} = \Delta u + \lambda u^3}$,

find A PRIORI ENERGY ESTIMATE.

Multiply → $\boxed{A u_t + B \cdot \nabla u + C u}$ and play with it!

~~MULTIPLY BY u → $\partial_t(\frac{1}{2}u^2) = \partial_i(\partial_i u \, u) - \partial_i u \, \partial_i u$~~

→ **Multiply PDE by u:**

$$\partial_t \left(\tfrac{1}{2} u^2\right) = \partial_i (\partial_i u \, u) - |\nabla u|^2 + \lambda |u|^4 ,$$

Integrate w.r.t. x: (u decays at ∞)

$$\partial_t \int \tfrac{1}{2} u^2 = \int \partial_i (\partial_i u \, u) - \int |\nabla u|^2 + \lambda \int |u|^4$$

Boundary at ∞

$$\Rightarrow \frac{d}{dt} \int \tfrac{1}{2} u^2 + \int (|\nabla u|^2 - \lambda |u|^4) = 0 \qquad \Rightarrow \ldots$$

→ **If $\lambda = -1$:**

$$\int \frac{1}{2} u^2(t)\, dx + \int_0^t \int |\nabla u|^2(t)\, dx\, dt + \int_0^t \int_\Omega |u|^4 = \frac{1}{2} \int |u_0|^2\, dx$$

A PRIORI ESTIMATE, NOT SUFFICIENT for GLOBAL EXISTENCE

→ **Multiply by u_t:**

$$u_t^2 = \partial_i \left(\partial_i u u_t \right) - \partial_t \frac{1}{2} |\nabla u|^2 + \frac{\lambda}{4} \partial_t u^4$$

$\lambda = -1$: **A PRIORI ESTIMATE**

$$\int \frac{|\nabla u(t)|^2}{2}\, dx + \frac{\lambda}{4} \int u^4(t)\, dx + \int_0^t \int |\partial_t u|^2\, dx\, dt = \int \frac{|\nabla u(0)|^2}{2}\, dx + \frac{\lambda}{4} \int u^4(0)\, dx.$$

→ These estimates NOT ENOUGH to get GLOBAL EXISTENCE in H^2 - PICARD

→ BUT can try to get H^1 - GLOBAL EXISTENCE THEORY!

(without assuming ODE theory).

- $u_t = \Delta u - u^3$, $x \in \mathbb{R}^3$
- A PRIORI ESTIMATES (by above):

$$|u(t)|_{H^1} \leq C\,|u_0|_{H^1}$$

$$|\nabla u|_{L^2_t L^2_x} \leq C\,|u_0|_{H^1}$$

$$|u_t|_{L^2_t L^2_x} \leq C\,|u_0|_{H^1}.$$

$$u(t) = e^{\Delta t} u_0 - \int_0^t e^{\Delta(t-s)} u^3(s)\, ds$$

Try CONTRACTION argument on H^1:

$$|u|_{H^1} \le c\,|u_0|_{H^1} + \int_0^t \left| e^{\Delta(t-s)} u^3(s) \right|_{H^1} ds$$

Look only at highest order derivatives. It's the same (can bound lower order):

$$|\nabla u|_{L^2} \le c\,|\nabla u_0|_{L^2} + \int_0^t \left| \nabla e^{\Delta(t-s)} u^3(s) \right|_{L^2} ds$$

$$\cancel{|\nabla(u^3)|_{L^2}}$$

$$\cancel{|u^2 \nabla u|_{L^2}}$$

Can sacrifice singularity in "t"

to get better estimates:

$$\left| e^{\Delta(t-s)} u^3(s) \right|_{H^1} \le \frac{1}{(t-s)^{1/2}} |u^3|_{L^2} \Rightarrow \cdots$$

$$= \frac{1}{|t-s|^{1/2}} |u|_{L^6}^3 \le \frac{1}{|t-s|^{1/2}} |\nabla u|_{L^2}^3$$

SOBOL. EMB. in $\mathbb{R}^3$
$|u|_{L^6} \le C |Du|_{L^2}$

... $\int_0^t \frac{1}{|t-s|^{1/2}} |Du(s)|_{L^2}^3 ds$, (CANNOT DO IT in L^4)

$\sqrt{t}\, |u|_{L^\infty H^1}^3$.

→ **Have Global solutions in H^1**

- Suppose $u_0 \in H^k$, $k \ge 1$.
Can I get H^k bound ?

EX: $k=2$:

$$|u|_{H^2} \le c |u_0|_{H^2} + \int_0^t \frac{1}{|t-s|^{1/2}} |u^3|_{H^1} ds$$

$|u^3|_{L^2}$ (V), $|u^2 Du|_{L^2}$

↓ Sobolev Emb.

$|u|_{L^6}^3 \; |u|_{H^2}$

$$\Rightarrow |u|_{H^2} \le c|u_0|_{H^2} + \int_0^t \frac{1}{|t-s|^{1/2}} |u(s)|_{H^2} ds$$

GRÖNWALL INEQ :

$$x(t) \le \underbrace{a + \lambda \int_0^t x(s)\, ds}_{z(t)}, \qquad a, \lambda > 0,$$

$$\to \quad \dot{z} = \lambda x \le \lambda z, \quad \frac{\dot{z}}{z} \le \lambda$$

$$\Rightarrow z(t) \le e^{\lambda t} z(0)$$

$$\to \quad x \le a + \int_0^t b(s) x(s)\, ds,$$

$$\Rightarrow \dot{z} = b x \le b z \Rightarrow \text{similar}$$

By Grönwall ineq; we get an H^2 bound. Then, we are on an algebra

$\Rightarrow$ Can get H^k bound!!

. REMARK : What matters is the a priori estimates.

Ex:

$$u_{tt} - \Delta u + u^3 = 0.$$

$H^1 \times L^2$ is our space.

($H^{k+1} \times H^k$ for linear problem.)

~~$\partial_t (u_t u) = |\partial_t u|^2 -$~~

→ Multiply by u : get nothing!!

→ Multiply by u_t :

$$\partial_t \frac{1}{2} |u_t|^2 - \partial_i (\partial_i u \, u_t) + \underbrace{\partial_i u \, \partial_i u_t}_{\partial_t \frac{1}{2} |\nabla u|^2} + \partial_t \frac{u^4}{4} = 0$$

$$\boxed{\partial_t \left(\frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 + u^4 \right) - \partial_i (\partial_i u \, u_t) = 0}$$

→ May need

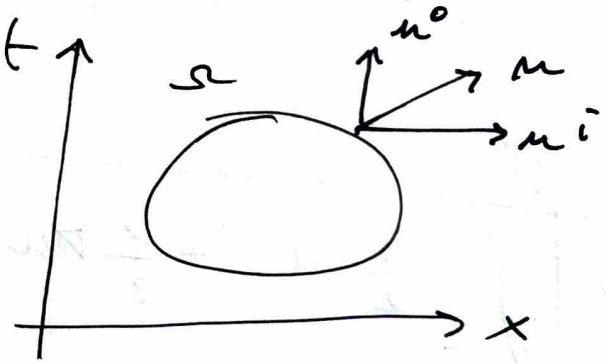
$$u(t) = u(0) + \int_0^t \partial_t u(s)\, ds$$

Integrate previous equality:

$$\int_{\partial\Omega} \left(n^0_t \left(\frac{1}{2} u_t^2 + \frac{1}{2}|\nabla u|^2 + u \right) - n^i_x \partial_i u\, u_t \right)$$

n^0_t: time component

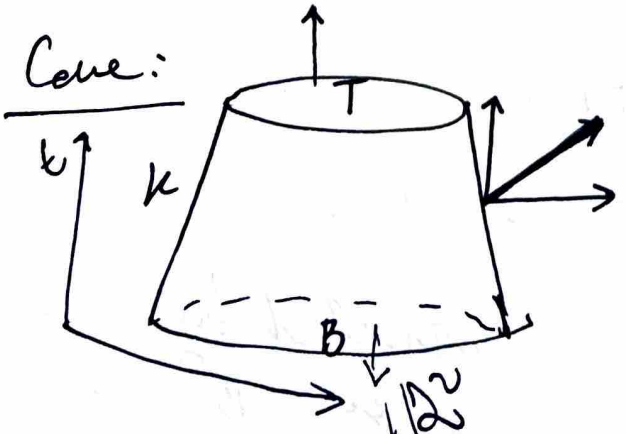
n^i_x: x-component



$\Rightarrow$

$\Rightarrow$ **Need $n^i < n^0$** so can complete the square.

Cone:



$$|x - x_0| = |t - t_0|$$

$$n = \frac{1}{\sqrt{2}} e_0, \frac{1}{\sqrt{2}} e_{\dots}$$

$$\int_T \frac{1}{2} u_t^2 + \frac{1}{2}|\nabla u|^2 + \frac{u^4}{4} +$$

$$+ \int_K \frac{1}{\sqrt{2}} \left(\frac{1}{2} u_t^2 + \frac{1}{2}|\nabla u|^2 + \frac{u^4}{4} \right) - \frac{1}{\sqrt{2}} e_n \cdot \nabla u u_t$$

$$- \int_B \frac{1}{2} u_t^2 + \frac{1}{2} \nabla u^2 + \frac{u^4}{4} = 0$$

$$\Rightarrow \boxed{\int_T \frac{1}{2} u_t^2 + \frac{1}{2} \nabla u^2 + \frac{u^4}{4} \leq \int_B \frac{1}{2} u_t^2 + \frac{1}{2}|\nabla u|^2 + \frac{u^4}{4}}$$

B = Time zero.

T = time t.

$\Rightarrow$ Quantity on T controlled by quantity on B (time zero).

NAVIER - STOKES EQN.

$$\partial_t u - \Delta u = \partial_x u^2 .$$

→ Global existence on $\mathbb{R}^2$ can be done.

→ ABC method:

$$\frac{1}{2}\int u^2(t)\,dx + \int_0^t \int |\nabla u(s)|^2\,dx\,ds = \int |u_0|^2\,dx$$

$$\left(\int u\,\partial_x u^2 = \int \partial_x \frac{u^3}{3} = 0. \right)$$

Drops out

$u_0 \in L^2 \Rightarrow \exists$ Global Solution.

Try as an exercise.

• SOBOLEV INTERP. INEQ :

$u : \mathbb{R}^d \longrightarrow \mathbb{R}$, $1 \le p, q \le \infty$,

$$\frac{1}{r} = \frac{j}{d} + \left(\frac{1}{p} - \frac{k}{d}\right)\alpha + \frac{1-\alpha}{q},$$

$$\frac{j}{d} \le \alpha \le 1. \text{ Then,}$$

$$|D^j u|_{L^r} \lesssim |D^k u|_{L^p}^{\alpha} \, |u|_{L^q}^{1-\alpha}$$

EX: $k > \frac{d}{p}$, $u \in W^{k,p}(\mathbb{R}^d)$.

→ let $j = 0$, $q = p$. Then,

$$\alpha = 0 \Rightarrow \frac{1}{r} = \frac{1}{p} > 0$$

$$\alpha = 1 \Rightarrow \frac{1}{r} = \frac{1}{p} - \frac{k}{d} < 0$$

→ $\exists \ \alpha \in (0,1)$ s.t. $\frac{1}{r} = 0$, i.e.,

$r = \infty$.

→ $\boxed{|u|_{L^\infty} \lesssim |D^k u|_{L^p}^{\alpha} \, |u|_{L^p}^{\alpha} < \infty}$, i.e., $u \in L^\infty$

Ex: $u \in W^{k,p}(\mathbb{R}^d)$, $k > \frac{d}{p}$,

$F \in C^N(\mathbb{R}, \mathbb{R})$, $N \geq k$.

Then,

(i): For any $m \in \mathbb{N}_+$,

$$\left| D^k(u)^m \right|_{L^p} \leq \left| u \right|^m_{W^{k,p}}$$

$$\left| (u)^m \right|_{W^{k,p}} \leq \left| u \right|^m_{W^{k,p}}$$

(ii): $$\left| F(u) \right|_{W^{k,p}} \leq \left(\left| u \right|_{W^{k,p}} + \left| u \right|^k_{W^{k,p}} \right)$$

↑ Constant depends on $|u|_{L^\infty}$.

Also, a more particular case is

$u, v \in W^{k,p}(\mathbb{R}^d)$, then for $k > \frac{d}{p}$:

$$\left| uv \right|_{W^{k,p}} \leq \left| u \right|_{W^{k,p}} \otimes \left| v \right|_{W^{k,p}}$$