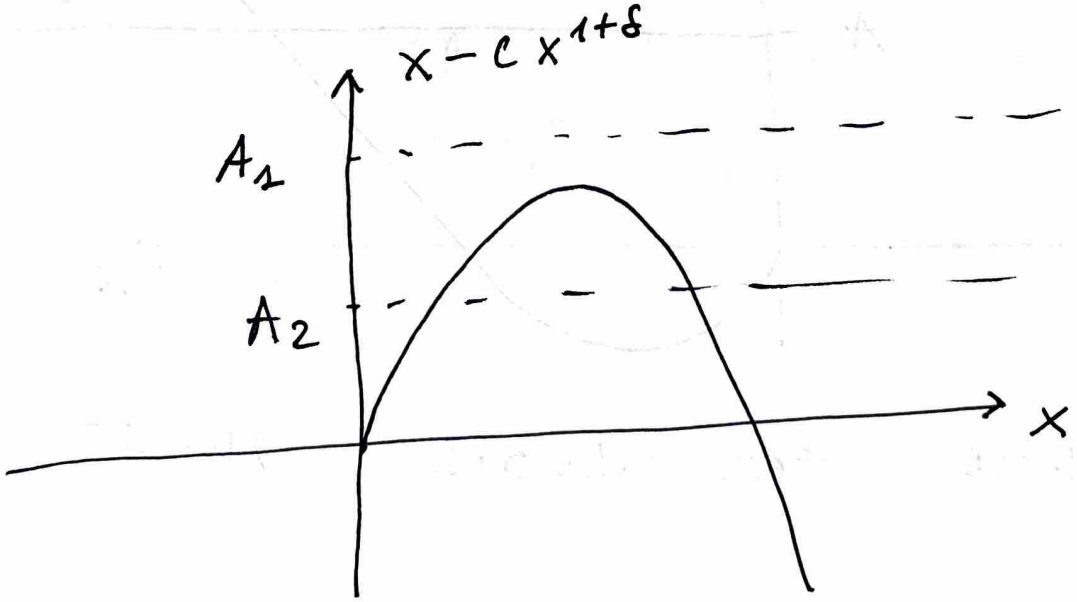
- **Class #5 after Midterm:**

- ENERGY ESTIMATES
- HYPERBOLIC EQUATIONS (DISPERSIVE)
- UNIQUENESS
- POTENTIAL THEORY.

- **ENERGY ESTIMATES:**

→ Always after inequality:

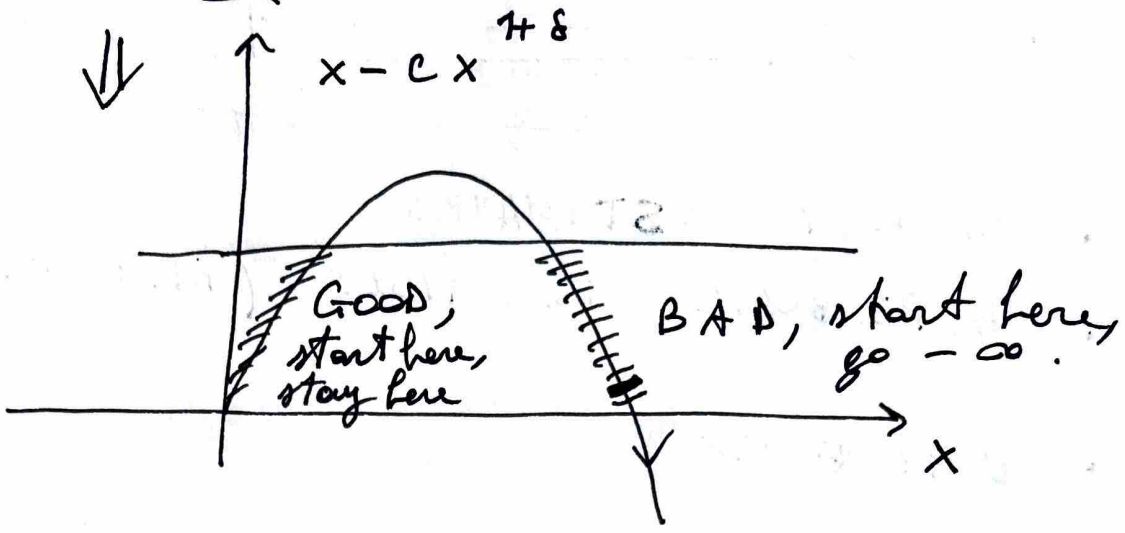
(i): $X \leq C X^{1+\delta} + A \qquad \delta > 0.$



- $A = A_2$, large $\Rightarrow$ USELESS!
- C large $\Rightarrow$ same USELESS!

• $A = A_2$ small <u>or</u> C small :

$\Downarrow$

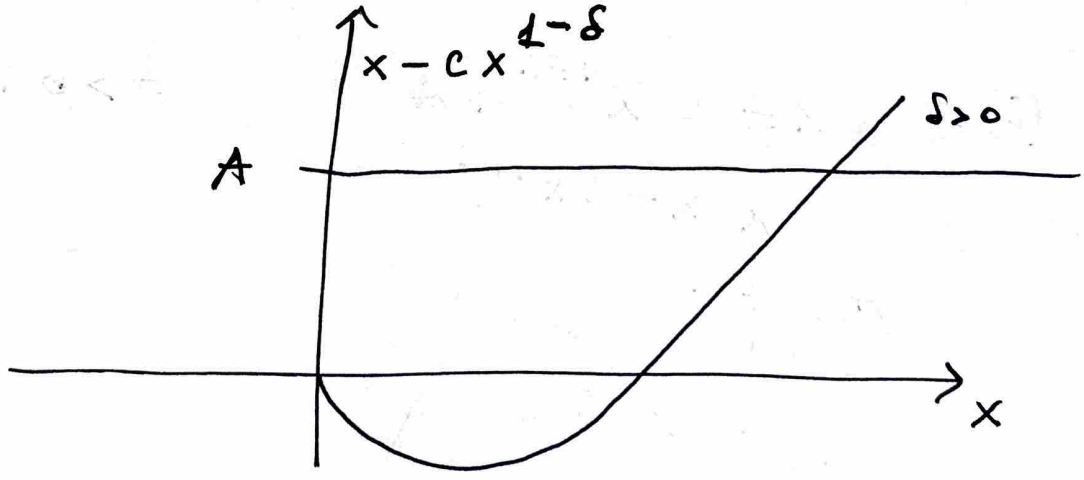


→ GOOD FOR LOCAL ESTIMATES

→ $a(t) = \int_0^T \int |\nabla u|^2 \, dx \, dt$, $T = 0 \Rightarrow a(T) = 0$
$\Rightarrow$ STAY in GOOD PART

→ $|u(t)| \leq T |u|_{L^2}^2 + A \Rightarrow$ choose T so small so that GOOD!

(ii) : $x \leq C x^{1-\delta} + A$, $\delta \geq 0$



→ GOOD FOR GLOBAL ESTIMATES.

- **UNIQUENESS**:

(HW #8)

$u_t = u_{xx} + f(u)$, on $\mathbb{R}$.

$$u^{n+1} = e^{\Delta t} u_0 + \int_0^t e^{\Delta(t-s)} f(u^n(s))\, ds$$

- A PRIORI ESTIMATES $\Leftrightarrow$ $\{u^n\}$ Bound:

$$\rightarrow |u^{n+1}|_{H^1} \le A|u_0|_{H^1} + \int_0^t \underbrace{|f(u^n(s))|_{H^1}}_{\le C_{|u^n|_{L^\infty}} |u^n|_{H^1}} ds$$

$$\left(|f(u)|_{L^2} + |\partial_x f(u)|_{L^2} \right.$$

$$f(u) \le C_u |u| \le C_{|u|_{L^\infty}} |u|$$

$$|u(x)^3| \le |u|_{L^\infty}^2 \quad u(x) \qquad \text{M.V.THM.}$$

$$\left. |f'(u)\, u_x|_{L^2} \le \text{same ...} \right)$$

$\rightarrow$ For A PRIORI ESTIMATES: (By above)

$$\|u\|_T = \sup_{t \in [0,T]} |u(t)|_{H^1} \qquad \Longrightarrow \infty$$

$$\|u\|_T \leq A + T\, C_{|u|_{L^\infty}} \|u\|_T .$$

→ Inequality (i). because of dependency on u.

(• Simpler Problem:

$$\|u\|_T \leq A + T\|u\|_T^2 ,$$

(Look at $x < 100 + Tx^2$,
Insure that you are in (i) GOOD
by taking T small !!)

For $T=0$, $\|u\|_T \leq A$,
want $\|u\|_T \leq 2A$ for
some small T to choose:

$$T\|u\|_T < \frac{1}{2}$$

$$2TA < \frac{1}{2}$$

$$\Rightarrow T < \frac{1}{4A}$$

$$\Rightarrow \|u\|_T \leq A + \frac{1}{2}\|u\|_T$$

$$\Rightarrow \frac{1}{2}\|u\|_T \leq A \Rightarrow \|u\|_T \leq 2A \;) \ldots$$

... In other words give me data of size A, can have uniform bound $\|u\|_T \leq 2A$ for $T < \frac{1}{4A}$.

→ Back to our Ineq.

$$\|u\|_T \leq A + T\, C_{|u|_{L^\infty}} \|u\|_T, \quad \cancel{(|u|_{L^\infty} \lesssim |u|_{H^1})}$$

→ Want bound $\|u\|_T \leq 10 \cdot A$

→ $T\, C_{10A} < \delta$, (*)

[where $C_{10A} = |f'(u)|_{L^\infty}$, $f \in C^1(\mathbb{R})$, $|u|_{L^\infty} \lesssim |u|_{H^1}$]

→ $(1-\delta)\|u\|_T \leq A$

→ $\boxed{\|u\|_T \leq \frac{A}{1-\delta} < 10A}$

for appropriate δ in (*).

A PRIORI ESTIMATE

⟹ DO ITERATION ...

$$\|u^{n+2}\|_T \leq A + T\, C_{|u^n|_{L^\infty}} \|u^n\|_T .$$

→ By APRIORI BOUND gotten before, for good choice of T:

$$\|u^n\|_T < 10A \Rightarrow \|u^{n+1}\| < 10A$$

(?)

→ APRIORI BOUND implies actual bound on sequence $\{u^n\}_{n=1}^{\infty}$.

→ Now want to show CONTRACTION (i.e., convergence):

$$\|u^{n+2} - u^{n+1}\|_T \leq T \|f(u^{n+1}) - f(u^{n-1})\|_T$$

$$+ \int_0^1 \left| f(u^{n+1}) - f(u^{n-1}) \right|^2 dx =$$

$$= \int \underbrace{\left| f(u^{n+1}) - f(u^{n-1}) \right|^2 dx}_{\{u^n\}_n \subset B \Rightarrow \text{MVT implies bound.}} +$$

$$\int \underbrace{\left| f'(u^n) u_x^n - f'(u^{n-1}) u_x^{n-1} \right|^2 dx}_{=}$$

$$f'(u^n)(u_x^n - u_x^{n-1}) + (f'(u^n) - f'(u^{n-1})) u_x^{n-1}$$

- $f'(u^n)\left(u_x^n - u_x^{n-2}\right)$ bounded
- $\left(f'(u^n) - f'(u^{n-1})\right) u_x^{n-1}$

CANNOT BOUND if $f \in C^1$ only!

→ NEED f' to be at least Lipschitz.

- **REMEDY:** GAIN REGULARITY in x at expense of $\frac{1}{t^{1/2}}$:

$$\left| e^{\Delta(t-s)} f(u^n) - f(u^{n-1}) \right|_{H^1}$$

$$\leq \frac{e}{\sqrt{(t-s)}} \left| f(u^n) - f(u^{n-2}) \right|_{L^2}$$

use M.V.T. to get bound ✓

& use $|w|_{L^2} \leq |w|_{H^1}$ ∎

- **ANOTHER REMEDY:**

Bound in H^1 and CONVERGENCE in L^2.

IDEA: Get BOUND in Strongest norm (H^1) & convergence in WEAKEST NORM. (L^2).

$$u_t = u_{xx} + f(u_x).$$

$$u_{tt} - \partial_x \left[\left(1 + \tfrac{1}{2}\cos(u)\right) u_x \right] = 0.$$

→ A priori Estimates by Integrating by parts (like standard wave eqn).

→ Convergence:

$$u_{tt}^{n+1} - \partial_x \left[\left(1 + \tfrac{1}{2}\cos(u^n)\right) u_x^{n+1} \right] = 0.$$

$$u_{tt} - u_{xx} + \varepsilon u_{xx}^2 = 0$$

→ Want Newton's Method:

$$u_{xtt} - u_{xxx} + 2\varepsilon u_{xx} u_{xxx} = 0$$

→ Now iteration is possible on L.o. derivative:

$$u_{xtt}^{n+1} - u_{xxx}^{n+1} + 2\varepsilon u_{xx}^{n} u_{xxx}^{n+1} = 0.$$

BUT DERIVATIVE is NOT INVERTIBLE

So NEED DIFFERENTIAL OPERATOR THAT IS INVERTIBLE: $(-\Delta + I)$:

$$(1 - \partial_x^2) u =: v$$

$$\Rightarrow \quad v_{tt}^{n+1} - v_{xx}^{n+1} + 2\varepsilon u_x^{n} v_{xx}^{n+1} - 2\varepsilon w_{xxx}^{n+2} = 0.$$

↳ Can iterate.

⇒ Use v to get a pseudo estimate in very high norm!

Thus, we showed UNIQUENESS by showing CONTRACTION on the space ...

ANOTHER WAY FOR UNIQUENESS:

$$u_{tt} - a(x,t)\, u_{xx} = 0 ,$$

ex: $1 + \frac{1}{2}\cos(x^2+t^2) = a(x,t)$.

a analytic, analytic data on $t=0$ non-characteristics.

$\Rightarrow$ C-K there is a local solution. (u_1, u_2 analytic sol $\Rightarrow$ C-K $\Rightarrow$!UNIQUENESS)

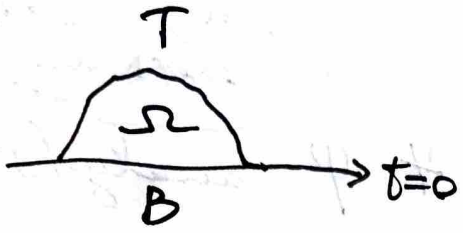
$\rightarrow$ Suppose u_1 analytic solution and u_2 $\neq 1$ solution.

$Lu = 0$, [Cauchy Problem]

u_1 analytic solution, $u_2 \neq 1$ solution (local).

$\Longrightarrow$ ...

→ $\varphi \in C_0^\infty$, $\int_{\mathbb{R}^d} (Lu)\varphi = \int_{\mathbb{R}^d} u L^* \varphi +$



$+ \int_B (\quad)$

$+ \int_T (\quad)$

→ Suppose u_1, u_2 2 solutions to CAUCHY PROBLEM ⇒ (any solution, also not ANALYTIC)

$$u|_B = (u_1 - u_2)|_B = 0$$

⇒ $$0 = \int_{\mathbb{R}^d} (Lu)\varphi = \int_{\mathbb{R}^d} u L^* \varphi + \cancel{\int_B (\quad)}^{\,0} + \int_T (\quad)$$

⇒ $$0 = \int_{\mathbb{R}^d} u L^* \phi + \int_T (\quad)$$

⇒ φ up to me to chose, thus

choose $\varphi|_T = 0$, $\partial^\alpha \varphi|_T = 0$

⇒ $\cancel{\int_T (\quad)}^{\,0}$ ⇒ $0 = \int u L^* \varphi$ ⇒ ...

Suppose thus can solve

$$\partial^\alpha \varphi\big|_\Gamma = 0 \quad , \qquad L^* \varphi = \psi \ , \ \forall\, \psi \text{ analytic}$$

$\alpha = 0, \ \alpha = 1, 2, \ldots$ BASED ON ORDER OF PDE.

$$0 = \int u \cdot \psi \quad , \text{ for any } \psi \text{ analytic}$$

$$\Rightarrow \quad u = 0 \, .$$

↑

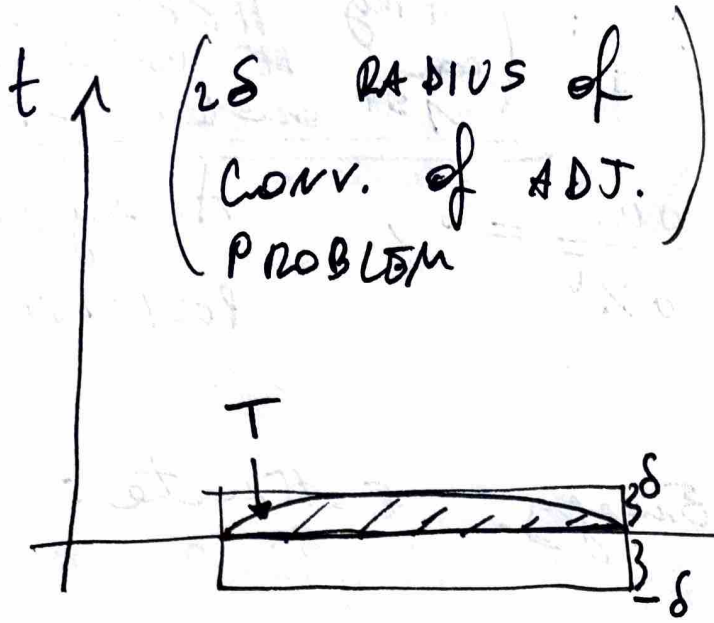
HOLMGREN UNIQUENESS THM:

↳ REQUIRES SOLVING ADJOINT EQUATION

↳ IF can Solve ADJ. EQN. locally by C-K, then HOLMGREN APPLIES!

↳ Useful for uniqueness of non-analytic solutions

(for u_1, u_2 analytic solutions UNIQUENESS by C-K.).



$Lu = 0$, coeffs.

$\partial^\alpha u\big| = f_\alpha$

$(0 \le \alpha \le n-1,$
$t = 0$

$L^* \varphi = \psi$, ψ analytic.

$\partial^\alpha \varphi\big|_T = 0$

$(0 \le \alpha \le n-1)$

- ~~Th C-K, convergence of analytic solution~~

HYPERBOLIC EQN. : (Any HYPERB. Eq. can be written as 1st order SYSTEM)

$$\frac{\partial u}{\partial t} - A_i \frac{\partial u}{\partial x^i} = 0, \qquad A \text{ symmetric}$$

POSITIVE DEFINITE

→ Can Get Energy estimate:

$$\left(\frac{\partial u}{\partial t}, u\right) - \left\langle A^i \frac{\partial u}{\partial x^i}, u \right\rangle$$

$$\int \left(A^i \partial_i u, u\right) dx = - \int \langle u, A^i \partial_i u \rangle - \int \langle \partial_i A^i u, u \rangle$$

$$\Rightarrow \int (A^i \partial_i u, u) = -\frac{1}{2} \int \left((\partial_i A^i) u, u\right) \leq C \int u^2.$$

$$\Rightarrow \boxed{\frac{d}{dt} \int \frac{1}{2} |u|^2 \leq \int |u|^2}$$

ENERGY ESTIMATE.

→ Differentiate eqn. once to get higher order estimates ...

- **Characteristics:**

Assume $\varphi(x,t) = c$ a charact. surface.

$\rightarrow \det\left(\varphi_t I - A_i \partial_i \varphi\right) = 0$

If A symmetric positive def, eigenvals all $\mathbb{R}$!

$\rightarrow \det\left|\tau - A_i \xi_i\right| = 0$ solve for

$\xi = \partial_i \varphi$ $\qquad \tau = \sigma(x,t,\xi)$

$\tau = \partial_t \varphi$.

eigenvalues, which are all $\mathbb{R}$, homogeneous of degree 1.