1. ELLIPTIC THEORY:

- Ω smooth, bounded open $\subset \mathbb{R}^n$, $u, f : \overline{\Omega} \longrightarrow \mathbb{R}$ smooth, consider:

$$\circledast \quad \begin{cases} -\Delta u = f & \text{on } \Omega. \\ u|_{\partial\Omega} = 0 \end{cases}$$

- Take $\phi \in C_c^\infty(\Omega)$, then INTEGRATION BY PARTS yields:

$$\textcircled{\Delta}: \int_\Omega Du \cdot D\phi \, dx = \int_\Omega f \phi \, dx, \quad \forall \phi \in C_c^\infty(\Omega).$$

- CONVERSELY: If f, Ω smooth and u any smooth solution to $\textcircled{\Delta}$, then

$$-\Delta u = f.$$

- Suppose $Du \in L^2(\Omega)$ and $\phi \in C_c^\infty(\Omega)$

  Then,

$$\int_\Omega Du \cdot D\phi \, dx \leq |Du|_{L^2} \, |D\phi|_{L^2} < \infty$$

- Suppose $f \in L^2(\Omega)$, $\phi \in C_c^\infty(\Omega)$. Then,

$$\int_\Omega f \phi \, dx \leq |f|_{L^2} \, |\phi|_{L^2}$$

- Thus, (△) is well-defined for $u \in H^1(\Omega)$, $f \in L^2(\Omega)$, $\phi \in C_c^\infty(\Omega)$.

- **NOTE:** 1- Can impose BDRY CONDITION $u|_{\partial\Omega} = 0$ by taking $u \in H_0^1(\Omega)$

  2- Can extend (△) by continuity to $\phi \in H_0^1(\Omega)$.

- In general, $|\langle f, \phi \rangle| \leq |f|_{H_0^1(\Omega)^*} |\phi|_{H_0^1(\Omega)}$ $\underline{< \infty}$ for any $f \in H_0^1(\Omega)^*$ (the DUAL SPACE of $H_0^1(\Omega)$).

- $\boxed{\text{DEF}}$: $H^{-1}(\Omega) := \left\{ f : H_0^1(\Omega) \to \mathbb{R},\ f \ \underline{\text{BDD}} \ \underline{\& \text{ LINEAR}} \right\}$

and the action of f on $\phi \in H_0^1(\Omega)$ denoted by $\langle f, \phi \rangle$. Norm of f given by

$$|f|_{H^{-1}(\Omega)} = \sup \left\{ \frac{|\langle f, \phi \rangle|}{|\phi|_{H_0^1}} : \phi \in H_0^1(\Omega), \phi \neq 0 \right\}$$

$f \in L^2(\Omega)$ defines $F_f \in H^{-1}(\Omega)$ by

$$\langle F_f, \phi \rangle := \int_\Omega f \phi \, dx, \quad \phi \in H_0^1(\Omega).$$

$$= (f, \phi)_{L^2}.$$

- $$|\langle F_f, \phi \rangle| = \left| \int_\Omega f \phi \, dx \right|$$

$$\leq |f|_{L^2} \, |\phi|_{L^2}$$

$$\leq |f|_{L^2} \, |\phi|_{H_0^1}$$

$\Rightarrow$ F_f is <u>bouded</u> on $H_0^1(\Omega)$ and clearly <u>linear</u>.

- $L^2(\Omega) \subset H^{-1}(\Omega)$, but it is not equal, as there are elements of H^{-1} that are not in L^2.

$\Rightarrow$

$$\boxed{\begin{array}{c} H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega) \\ \text{with} \quad H_0^1(\Omega)^* = H^{-1}(\Omega) \end{array}}$$

- Moreover, if $f \in L^2(\Omega) \subset H^{-1}(\Omega)$, $\phi \in H_0^1(\Omega)$

$$\langle f, \phi \rangle = (f, \phi)_{L^2}.$$

- **DEF**: Ω bdd, open $\subset \mathbb{R}^n$, $f \in H^{-1}(\Omega)$.
A function $u : \Omega \longrightarrow \mathbb{R}$ is a
WEAK SOLUTION to (*) if

(i): $u \in H_0^1(\Omega)$

(ii): $\int_\Omega Du \cdot D\phi \, dx = \langle f, \phi \rangle$,
$\forall \ \phi \in H_0^1(\Omega)$.

- NON-HOMOGENEOUS DIRICHLET PROBLEM:

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ u|_{\partial\Omega} = g \end{cases}$$

- (1): Can take $w \in H^1(\Omega)$ s.t. $w|_{\partial\Omega} = g$
and just replace (i) in DEF above
with $\tilde{u} := u - w \in H_0^1(\Omega)$, where
$\tilde{u}$ is a WEAK SOLUTION to

$$\begin{cases} -\Delta \tilde{u} = \tilde{f} & \text{on } \Omega \\ \tilde{u}|_{\partial\Omega} = 0 \end{cases}, \quad \left(\tilde{f} = f + \Delta w\right).$$

• POINCARÉ' INEQ. on $H_0^1(\Omega)$:

• let $u \in C_c^\infty(\Omega)$, Ω bounded, open $\subset \mathbb{R}^n$.
Then,

$$u(x_1, x') = \int_{-\infty}^{x_1} \partial_1 u(s, x')\, ds$$

$$\rightsquigarrow |u(x_1, x')| \leq \int_{-\infty}^{+\infty} |\partial_1 u(s, x')|\, ds$$

$$= \int_{\Omega_1} |\partial_1 u(s, x')|\, ds$$

$$\leq |\Omega_1|^{1/2} \left(\int_{\Omega_1} |\partial_1 u(s, x')|^2 ds \right)^{\frac{1}{2}}$$

$$\rightsquigarrow |u(x_1, x')|^2 \leq |\Omega_1| \int_{\Omega_1} |\partial_1 u(s, x')|^2 ds$$

$$\rightsquigarrow \int_{\Omega_1} |u(x_1, x')|^2 dx_1 \leq |\Omega_1|^2 \int_{\Omega_1} |\partial_1 u(s, x')|^2 ds$$

$\Longrightarrow \infty$

$$\dots \int_{\Omega'} \int_{\Omega_1} |u(x_1, x')|^2 \, dx_1 \, dx' =$$

$$= \int_{\Omega} |u(x)|^2 \, dx \leq |\Omega_1|^2 \int_{\Omega'} \int_{\Omega_1} |\partial_1 u(s, x')|^2 \, ds \, dx'$$

$$= |\Omega_1|^2 \int_{\Omega} |\partial_1 u(x)|^2 \, dx$$

$$\leq |\Omega_1|^2 \int_{\Omega} |Du(x)|^2 \, dx$$

$\rightsquigarrow$ By density, this holds for any $u \in H_0^1(\Omega)$

$\rightsquigarrow$ **THM**: let $u \in H_0^1(\Omega)$, Ω open $\subset \mathbb{R}^n$, Ω bounded. Then, $\exists$ C s.t.

$$\int_{\Omega} |u|^2 \, dx \leq C \int_{\Omega} |Du|^2 \, dx.$$

• **COROLLARY**: Ω open & bdd in $\mathbb{R}^n$, then $H_0^1(\Omega)$ equipped with inner product

$$(u,v)_0 := \int_\Omega Du \cdot Dv \, dx$$

is a HILBERT SPACE, and corresponding norm is $\simeq$ to standard norm in $H_0^1(\Omega)$.

PROOF:

$$|u|_0 := \left(\int_\Omega |Du|^2 \, dx \right)^{1/2}$$

$$|u|_1 := \left(\int_\Omega |u|^2 + |Du|^2 \, dx \right)^{1/2}$$

$$(u,v)_1 = \int_\Omega uv + Du \cdot Dv \, dx$$

By POINCARÉ',

$$|u|_0 \leq |u|_1 \leq (C+1)^{1/2} |u|_0 \quad \blacksquare$$

• EXISTENCE OF WEAK SOLUTIONS TO DIRICHLET PROBLEM :

(1): $f \in H^{-1}(\Omega)$, then $\exists!$ WEAK SOLUTION $u \in H_0^1(\Omega)$ to

$$-\Delta u = f$$

$$\left(u|_{\partial\Omega} = 0 \right).$$

PROOF: $v \in \left(H_0^1(\Omega), |\cdot|_0\right)$

$$\int Du \cdot Dv \, dx = \langle f, v \rangle .$$

By RIESZ REPR., $\exists!$ $u \in H_0^1$ s.t.

$$\langle f, v \rangle = (u, v)_0 = \int_\Omega Du \cdot Dv ,$$

$\forall$ $v \in H_0^1(\Omega)$. ∎

(2): $f \in H^{-1}(\Omega)$, consider

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 . \end{cases}$$

$\rightsquigarrow$ u is a WEAK SOL. if $u \in H_0^1(\Omega)$ &.

$$\int_\Omega (Du \cdot Dv + uv)\, dx = \langle f, v \rangle ,$$

$\forall\ v \in H_0^1(\Omega)$.

$\rightsquigarrow$ By RIESZ REPR. w.r.t. $\exists!\ u \in H_0^1(\Omega)$

$$\langle f, v \rangle = (u, v)_1$$

$$= \int_\Omega (Du \cdot Dv + uv)\, dx .$$

(we equipped $H_0^1(\Omega)$ with standard inner product). ∎

In (2), we did NOT use POINCARÉ' inequality $\Rightarrow$ Ω can be $= \mathbb{R}^n$

$\rightsquigarrow$ Get $\exists!\ u \in H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n)$ for any $f \in H^{-1}(\mathbb{R}^n)$. Further,

$$|f|_{H^{-1}} = |u|_{H^1} \quad \text{by duality}$$

$\Rightarrow$ $(-\Delta + I)^{-1} : H^1(\mathbb{R}^n) \longrightarrow H^{-1}(\mathbb{R}^n)$ is an ONTO-ISOMETRY between $H^1(\mathbb{R}^n)$ & $H^{-1}(\mathbb{R}^n)$.

(3): $\mu > 0$, $f \in H^{-1}(\Omega)$ - consider

$$\begin{cases} -\Delta u + \mu u = f & \text{on } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

$\rightsquigarrow$ $u \in H_0^1(\Omega)$ is a weak sol. of

$$(u, v)_\mu := \int (Du \cdot Dv + \mu u v)\, dx$$

$$= \langle f, v \rangle, \quad \forall\ v \in H_0^1(\Omega).$$

$\rightsquigarrow$ let $c = \max\left\{\mu, \frac{1}{\mu}\right\}$. Then,

$$\frac{1}{c} |u|_\mu^2 \leq |u|_1^2 \leq c\, |u|_\mu^2$$

$\rightsquigarrow$ $(\cdot,\cdot)_\mu \approx (\cdot,\cdot)_1$ and again by RIESZ REPR., $\exists!\ u \in H_0^1(\Omega)$ (equipped w/ $(\cdot,\cdot)_\mu$) s.t.

$$(u, v)_\mu = \langle f, v \rangle, \quad \forall\ v \in H_0^1(\Omega). \quad \blacksquare$$

(4): • $\begin{cases} -\sum_{i,j} \partial_i \left(a_{ij} \partial_j u \right) = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$

$f \in H^{-1}(\Omega)$, $a_{ij} = a_{ji} \in L^\infty(\Omega)$

and satisfy <u>UNIFORM ELLIPTICITY CONDITION</u>: $\exists\ \theta > 0$ s.t.

$$\sum_{i,j} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

(Means for any fixed $x \in \Omega$, the symmetric matrix $A(x) = \left(a_{ij}(x)\right)_{ij}$ has all positive eigenvalues $\geq \theta$).

• Assume Ω bounded. Then a weak formulation is that $u \in H_0^1(\Omega)$ and

$$a(u,v) = \langle f, v \rangle, \quad \forall\ v \in H_0^1(\Omega)$$

⟶ ∞

- where $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is symmetric, bilinear form

$$a(u,v) := \sum_{ij} \int_\Omega a_{ij}\, \partial_j u\, \partial_i v .$$

→ Boundedness of a_{ij}, uniform ellipticity, Poincaré' Ineq. implies that $a(\cdot,\cdot)$ defines an INNER PRODUCT on $H_0^1(\Omega)$ equivalent to $(\cdot,\cdot)_1$.

→ RIESZ REPRESENTATION $\Rightarrow \exists!\, u \in H_0^1(\Omega)$ s.t.

$$a(u,v) = \langle f, v \rangle , \quad \forall\, v \in H_0^1(\Omega).$$

■

- Consider $Lu = f$,

$$Lu := -\sum_{i,j} \partial_i (a_{ij} \partial_j u) + \sum_i \partial_i (b_i u) + cu$$

where Ω open, $u : \Omega \to \mathbb{R}$.

- Assume $a_{ij}, b_i, c : \Omega \to \mathbb{R}$ all in $L^\infty(\mathbb{R}^n)$, $a_{ij} = a_{ji}$

- L is ELLIPTIC if $A(x) = (a_{ij}(x))$ is POSITIVE DEFINITE

- We assume L is UNIFORMLY ELLIPTIC on Ω: $\exists \ \theta > 0$ s.t.

$$\sum_{i,j} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$$

for A.E. $x \in \Omega$, $\forall \ \xi \in \mathbb{R}^n$.

- UNIFORM ELLIPTICITY CONDITION allows estimate of integral of $|Du|^2$ in terms of integral of $\sum_{ij} a_{ij} \partial_j u \partial_i$

- <u>EX:</u> $L = -\Delta$ uniformly elliptic with $\theta = 1$, on any open set, since $a_{ij} = \delta_{ij}$.

- <u>EX:</u> Tricomi Operator

$$L = -\left(y \partial_x^2 + \partial_y^2\right)$$

$\Rightarrow$ $y \xi_1^2 + \xi_2^2 > 0$ for $y > 0$, $|\xi| \neq 0$

$\Rightarrow$ • L is ELLIPTIC for $y > 0$ and HYPERBOLIC for $y < 0$.

- L is UNIFORMLY ELLIPTIC on the strip $\{(x,y) : \epsilon < y < 1, \ \epsilon \in (0,1)\}$.

- Consider DIR. PROB. for $L + \mu I$:

$$\circledast \begin{cases} Lu + \mu u = f & \text{on } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $Lu = -\sum_{i,j} \partial_i(a_{ij}\partial_j u) + \sum_i \partial_i(b_i u) + cu$.

- Take $\phi \in C_c^\infty(\Omega)$ and INTEGRATE BY PARTS:

$$\int_\Omega (Lu + \mu u)\phi \, dx = \int_\Omega f\phi \, dx$$

$$\Downarrow$$

$$\int_\Omega \left(\sum_{ij} a_{ij}\partial_j u \partial_i \phi - \sum_i b_i u \partial_i \phi + cu\phi \right) dx + \mu(u, \phi)_{L^2} = \langle f, \phi \rangle .$$

- Thus, $u \in H_0^1(\Omega)$ is a weak solution to $\circledast$ if

$$a(u,v) + \mu(u,v)_{L^2} = \langle f, v \rangle ,$$

$\forall\ v \in H_0^1(\Omega)$ and $a : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}$ given by bilinear form

$$a(u,v) = \int_\Omega \left(\sum_{ij} a_{ij}\partial_j u \partial_i v - \sum_i b_i u \partial_i v + c\, uv \right) dx .$$

- The form $a(\cdot,\cdot)$ IS NOT SYMMETRIC unless $b\vec{v} = 0$, $\forall \vec{v} \Rightarrow$ CANNOT DEFINE EQUIVALENT INNER PRODUCT to $(\cdot,\cdot)_1$ to use RIESZ REPRESENTATION.

- **THM**: (LAX - MILGRAM).

Let H be a Hilbert space, with INNER PRODUCT $(\cdot,\cdot) : H \times H \longrightarrow \mathbb{R}$. Let $a(\cdot,\cdot)$ be a BILINEAR FORM $a : H \times H \longrightarrow \mathbb{R}$. Suppose $\exists$ constants $c_1, c_2 > 0$ s.t.

$$c_1 \|u\|^2 \leq a(u,u), \quad |a(u,v)| \leq c_2 \|u\| \, \|v\|$$

$\forall u, v \in H$. Then, for every BOUNDED LINEAR FUNCTIONAL $f : H \longrightarrow \mathbb{R}$ $\exists!$ $u \in H$ s.t.

$$\langle f, v \rangle = a(u,v), \quad \forall v \in H.$$

• Ω OPEN, BOUNDED $\subset \mathbb{R}^n$.

• Consider general ELLIPTIC bilinear form as above:

□

$$a(u,v) = \int_\Omega \left\{ \sum_{ij} a_{ij} \partial_j u \partial_i v - \sum_i b_i u \partial_i v + c u v \right\} dx$$

where (i): $a_{ij} = a_{ji}$, $b_i, c \in L^\infty(\Omega)$

(ii): $\sum_{i,j} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$, $\theta > 0$

for a.e. x and all $\xi \in \mathbb{R}^n$.

• $|a(u,v)| \leq \sum_{i,j} \int_\Omega |a_{ij} \partial_j u \partial_i v| dx + \sum_i \int_\Omega |b_i u \partial_i v| dx$

$$+ \int_\Omega |c u v| dx$$

$$\leq \sum_{i,j} |a_{ij}|_{L^\infty} |\partial_j u|_{L^2} |\partial_i v|_{L^2} +$$

$$+ \sum_i |b_i|_{L^\infty} |u|_{L^2} |\partial_i v|_{L^2} +$$

$$+ |c|_{L^\infty} |u|_{L^2} |v|_{L^2}$$

$$\leq \left(\sum_{ij} |a_{ij}|_{L^\infty} + \sum_i |b_i|_{L^\infty} + |c|_{L^\infty} \right) |u|_{H^1_0} |v|_{H^1_0}$$

■

- By UNIFORM ELLIPTICITY:

$$\theta |Du|_{L^2}^2 = \theta \int_\Omega |Du|^2 dx$$

$$\leq \sum_{ij} \int_\Omega a_{ij} \partial_i u \partial_j u \, dx$$

$$= a(u,u) + \sum_i \int_\Omega b_i u \partial_i u \, dx - \int_\Omega c u^2 dx$$

$$\leq a(u,u) + \sum_i |b_i|_{L^\infty} \int_\Omega |u||\partial_i u| dx + |c|_{L^\infty} \int_\Omega |u|^2 dx \quad \cdots$$

By Cauchy with $\varepsilon > 0$:

$$\int_\Omega |u||\partial_i u| dx \leq \int_\Omega |u||Du| dx \leq \frac{1}{4\varepsilon} \int_\Omega |u|^2 dx + \varepsilon \int_\Omega |Du|^2 dx$$

Now choose $\varepsilon > 0$ so small so that

$$\varepsilon \beta < \frac{\theta}{2}, \quad \beta := \sum_i |b_i|_{L^\infty} \Longrightarrow \cdots$$

... We estimated:

$$\theta \left| Du \right|_{L^2}^2 \leq a(u,u) + \beta\varepsilon \left| Du \right|_{L^2}^2 + \frac{\beta}{4\varepsilon} \left| u \right|_{L^2}^2 + \left| c \right|_{L^\infty} \left| u \right|_{L^2}^2$$

where $\beta\varepsilon < \frac{\theta}{2}$. Thus,

$$\left(\frac{\theta}{2} - \beta\right) \left| Du \right|_{L^2}^2 \leq a(u,u) + C \left| u \right|_{L^2}^2 .$$

Ω bounded $\Rightarrow$ By <u>POINCARÉ'</u> on $H_0^1(\Omega)$:

$$\left| u \right|_{L^2(\Omega)}^2 \leq C \left| Du \right|_{L^2(\Omega)}^2$$

Thus, it follows that

$$\tilde{\beta} \left| u \right|_{H_0^1}^2 \leq a(u,u) + \gamma \left| u \right|_{L^2}^2$$

for $\tilde{\beta} > 0$, $\gamma \geq 0$ appropriate constants.

$\Longrightarrow$

- **THM**: Under assumptions above on a, $\exists\ \beta, \alpha > 0$ and $\gamma \geq 0$ s.t.

$$|a(u,v)| \leq \alpha |u|_{H_0^1} |v|_{H_0^1}$$

and

$$\beta |u|_{H_0^1}^2 \leq a(u,u) + \gamma |u|_{L^2}^2$$

(where $u, v \in H_0^1(\Omega)$).

- *Notice*: If $\gamma > 0$, Lax-Milgram assumptions are NOT SATISFIED !!

$\Rightarrow \dots$

- **THM**: (EXISTENCE THM)

$\exists$ number $\gamma \geq 0$ s.t. for each

$$\mu \geq \gamma$$

and each $f \in H^{-1}(\Omega)$ there exists a unique WEAK SOLUTION $u \in H_0^1(\Omega)$ to

$$\begin{cases} Lu + \mu u = f & \text{on } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

PROOF: $\mu \in \mathbb{R}$, $\mu \geq \gamma$ and define

$$a_\mu(u,v) = a(u,v) + \mu (u,v)_{L^2},$$

$u, v \in H_0^1(\Omega)$. Then, by previous estimates

- $$|a_\mu(u,v)| \leq \alpha |u|_{H_0^1} |v|_{H_0^1} + |\mu| |u|_{L^2} |v|_{L^2}$$
$$\leq (\alpha + |\mu|) |u|_{H_0^1} |v|_{H_0^1}.$$

$$\cdot\ \beta |u|^2_{H^1_0} \le a(u,u) + \gamma |u|^2_{L^2}$$

$$\le a(u,u) + \mu |u|^2_{L^2}$$

$$= a_\mu(u,u), \quad (\text{since } \mu \ge \gamma).$$

Thus, by LAX-MILGRAM, for each $f \in H^{-1}(\Omega)$, $\exists!\ u \in H^1_0(\Omega)$ s.t.

$$\langle f, v \rangle = a_\mu(u,v), \quad \forall\ v \in H^1_0(\Omega).$$

■

INTERIOR REGULARITY:

- Consider $Lu = -\sum_{i,j} \partial_i (a_{ij} \partial_j u)$ and the PDE:

$$\text{(A)} \qquad Lu = f \quad \text{on } \Omega,$$

where $f \in L^2(\Omega)$ and Ω open, bdd $\subset \mathbb{R}^n$, $a_{ij} \in C^1(\bar{\Omega})$.

- A function $u \in H^1(\Omega)$ is a weak solution to (A) if:

$$a(u,v) = \langle f, v \rangle, \quad \forall\, v \in H_0^1(\Omega),$$

where the bilinear form a is given by

$$a(u,v) := \sum_{i,j} \int_\Omega a_{ij} \partial_j u \, \partial_i v \, dx.$$

(i): We DO NOT IMPOSE any boundary conditions
→ INTERIOR REGULARITY THM applies to any weak solution of (A)!!

(ii): INTERIOR REGULARITY tells us $u \in H^2_{loc}(\Omega)$. Thus, $\forall\, v \in C_c^\infty(\Omega)$:

$$a(u,v) = \langle f, v \rangle = (f, v)_{L^2}$$

(i.e., u is a weak solution to (△)).

Since $u \in H^2_{loc}(\Omega)$, then can integrate by parts in $a(u,v)$ to bring 2nd derivatives back to u:

$$a(u,v) = \int_\Omega (Lu)\, v \, dx = (Lu, v)_{L^2}$$

$$= (f, v)_{L^2}$$

$$\Rightarrow \quad (Lu - f, v)_{L^2} = 0, \quad \forall\, v \in C_c^\infty(\Omega)$$

$$\Rightarrow \quad \boxed{Lu = f, \text{ A.E. } x \in \Omega.}$$

Suppose $u \in C_c^\infty(\Omega)$, then

$$\begin{aligned}
\int_\Omega (\Delta u)^2 \, dx &= \sum_{i,j} \int_\Omega \partial_i^2 u \, \partial_j^2 u \, dx \\
&= -\sum_{i,j} \int_\Omega \partial_i (\partial_i \partial_j u)(\partial_j u) \, dx \\
&= \sum_{i,j} \int_\Omega (\partial_i \partial_j u)(\partial_i \partial_j u) \, dx \\
&= \sum_{i,j} \int_\Omega (\partial_i \partial_j u)^2 \, dx \\
&= \int_\Omega |D^2 u|^2 \, dx
\end{aligned}$$

$\Rightarrow$ If $-\Delta u = f$, then

$$|D^2 u|_{L^2} = |f|_{L^2}$$

$\Rightarrow$ We "should" have $u \in H^2_{loc}(\Omega)$ if $f \in L^2(\Omega)$. But above calculations not justified for $u \in H^1(\Omega)$.

- **THM**: • Ω open, bdd set in $\mathbb{R}^n$.
  $a_{ij} \in C^1(\Omega)$, $f \in L^2(\Omega)$.
  • If $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ on Ω, then $u \in H^2_{loc}(\Omega)$.
  Further, for $\Omega' \subset\subset \Omega$:

$$|u|_{H^2(\Omega')} \leq C\left(|f|_{L^2(\Omega)} + |u|_{L^2(\Omega)}\right).$$

(Constant $C = C(n, \Omega, \Omega', a_{ij})$).

- **THM**: • Ω open, bdd. set in $\mathbb{R}^n$,
  $a_{ij} \in C^{k+1}(\Omega)$ and $f \in H^k(\Omega)$.
  • If $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ on Ω, then $u \in H^{k+2}_{loc}(\Omega)$.
  Further, for $\Omega' \subset\subset \Omega$:

$$|u|_{H^{k+2}(\Omega')} \leq C\left(|f|_{H^k(\Omega)} + |u|_{L^2(\Omega)}\right)$$

Constant $C = C(n, k, \Omega, \Omega', a_{ij})$.

REMARK: If above THM holds with $k > \frac{n}{2}$, then

$f \in C(\Omega)$ and $u \in C^2(\Omega)$

$\Rightarrow$ u IS a CLASSICAL SOLUTION of $Lu = f$. (on Ω).

- **THM:**
  - Ω open, bdd set in $\mathbb{R}^n$, $a_{ij}, f \in C^\infty(\Omega)$.
  - If $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ on Ω, then $u \in C^\infty(\Omega)$.

PROOF: By last THM, since $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ on Ω and $f \in H^k(\Omega)$, $a_{ij} \in C^{k+2}(\Omega)$ for all k's, then $u \in H^{k+2}_{loc}(\Omega)$ for all k's $\Longrightarrow$ ...

... i.e., for any $\Omega' \subset\subset \Omega$,

$u \in H^{k+2}_{\bullet}(\Omega')$, $\forall$ k's.

$\Rightarrow$ $u \in C^{\infty}(\Omega')$, $\forall$ $\Omega' \subset\subset \Omega$.

$\Rightarrow$ $u \in C^{\infty}(\Omega)$.

- **BOUNDARY REGULARITY** :

THM :

- Ω bounded open set in $\mathbb{R}^n$ with C^2 - boundary.
- $a_{ij} \in C^1(\bar{\Omega})$, $f \in L^2(\Omega)$.
- If $u \in H_0^1(\Omega)$ is a weak solution to $Lu = f$ on Ω, then $u \in H^2(\Omega)$ and

$$|u|_{H^2(\Omega)} \leq C\left(|f|_{L^2(\Omega)} + |u|_{L^2(\Omega)}\right)$$

$$C = C(n, \Omega, a_{ij})$$

PROOF: Idea is that can use partition of unity, flattening of boundary and interior regularity THM to estimate tangential derivatives up to the boundary (flat part).

$\Rightarrow$ Use PDE (in weak form) to conclude the transversal derivative is in L^2 ■

- **THM**:
  - Ω is a bdd, open set in $\mathbb{R}^n$, $\partial\Omega$ is C^{k+2}.
  - $a_{ij} \in C^{k+1}(\bar{\Omega})$, $f \in H^k(\Omega)$.
  - If $u \in H_0^1(\Omega)$ is a weak solution to $Lu = f$ on Ω, then $u \in H^{k+2}(\Omega)$ and

$$|u|_{H^{k+2}(\Omega)} \leq C \left(|f|_{H^k(\Omega)} + |u|_{L^2(\Omega)} \right)$$

$$C = C(n, k, \Omega, a_{ij})$$

$\Rightarrow$ • If $a_{ij}, f \in C^\infty(\bar{\Omega})$ and $\partial\Omega \in C^\infty$ and u a weak solution to $Lu = f$ on $\Omega \Rightarrow u \in C^\infty(\bar{\Omega})$.