EXISTENCE FOR AUTONOMOUS ODE's

$$\begin{cases} \dot{x} = F(x) \\ x(0) = x_0 \end{cases}, \quad x \in \mathbb{R}^d, \quad F \text{ LOC. LIP.}$$

- F LOC. LIPSCHITZ $\iff$ F LIPSCHITZ on COMPACT SET.
- Consider $C([0,T], \overline{B_2(x_0)})$, define

$$\phi(x)(t) =: x_0 + \int_0^t F(x(s))\, ds.$$

$$\rightarrow \quad |\phi(x_0) - x_0| \leq \int_0^t |F(x(s))|\, ds$$

$$\leq TM, \quad \text{where}$$

$$M := \sup_{x \in \overline{B_2(x_0)}} |F(x)|.$$

$\rightarrow$ For example, choose T so that

$$TM \leq 1.$$

$\rightarrow$ By continuity of F and by F.T.C.,

$$\phi : C([0,T], \mathbb{R}^d) \longrightarrow C([0,T], \mathbb{R}^d).$$

- let $x^{n+1}(t) = \phi(x^n)(t)$.

$$\rightarrow \quad |x^{n+1} - x^n|_C = \sup_{t \in [0,T]} |x^{n+1}(t) - x^n(t)|$$

$$\leq \int_0^T L \cdot |x^n - x^{n-1}|_C \, ds$$

$$= TL \, |x^n - x^{n-1}|_C,$$

where L is Lipschitz constant for F on $\overline{B_2(x_0)}$ $\rightarrow$ Need to show that $\{x^n\}_n \subset \overline{B_2(x_0)} \quad \forall \; t \in [0,T]$, by appropriately choosing T.

$\rightarrow$ let $TL := q$. Then, for $q \in (0,1)$:

$$|x^{n+1} - x_0|_C \leq \sum_{k=0}^{n} |x^{k+1} - x^k|_C$$

$$\leq \sum_{k=0}^{n} q^k \, |x^1 - x^0|_C$$

$$\leq \frac{1 - q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

$$\leq 2 \quad \text{for} \quad q < \frac{1}{2}.$$

$$\Rightarrow \{x^n\}_n \in \overline{B_2(x_0)}, \; \forall n.$$

→ Thus, we showed

(i): $\phi : C\left([0,T], \overline{B_2(x_0)}\right) \longrightarrow C\left([0,T], \overline{B_2(x_0)}\right)$

for $T < \frac{1}{2L}$

(ii): $C\left([0,T], \overline{B_2(x_0)}\right)$ is a CLOSED subset of Banach Space $C([0,T], \mathbb{R}^d)$, thus $C([0,T], \overline{B_2(x_0)})$ is also COMPLETE!

(iii): ϕ is a contraction on $C\left([0,T], B_2(x_0)\right)$

⇒ By Banach Fixed Point THM, $\exists!$ FIXED POINT $x(t)$ in $C([0,T], B_2(x_0))$, which clearly satisfies ODE from $[0,T]$. ■

- **REMINDER**: LOCALLY LIPSCHITZ ⇕ LIPSCHITZ on COMPACT SETS.

(*) $\begin{cases} \dot{x} = F(x) \\ x(T) = x_T \end{cases}$, $x \in \mathbb{R}^d$, F LOC. LIP., suppose $|x(t)| \le C(x_0)$

→ By previous argument $\exists!$ sol. on $[0, T]$.

→ Since $|x(t)| \le C(x_0)$, $\forall\, t \ge 0$, could define for $t \ge T$:

$$\phi(x)(t) := x_T + \int_T^t F(x(s))\, ds$$

& REPEAT PICARD'S ITERATION (+ Banach Fixed pt theorem) by using the fact that F is LIPSCHITZ on COMPACT SETS ($|x(t)| \le C(x_0)$) to get existence & UNIQUENESS on ...

... $[T, T+T^1]$, where T^1 is sufficiently small to get contraction and $\phi(X) \subseteq X$ in order to apply Banach. ■

(Choice of T^1 independent of T by UNIFORM ESTIMATE for $|x(t)|$)

$\Rightarrow \quad T = \infty$. ■

GLOBAL EXISTENCE / CONTINUATION ARGUMENT:

- $T_+ = \sup \{ t > t_0 \mid \exists$ a solution to IVP on $[t_0, t) \}$

IVP: $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$,

f Locally Lipschitz on D open,
$D \subset \mathbb{R} \times \mathbb{R}^n$.

- **THM**: Suppose $f \in (C, \text{Lip}_{loc})$ on open set $D \subset \mathbb{R} \times \mathbb{R}^n$. Let $(t_0, x_0) \in D$ and (T_-, T_+) be MAXIMAL INTERVAL of EXISTENCE ($T_+ < \infty$) to IVP $\dot{x} = f(t,x)$, $x(t_0) = x_0$.

Given compact $K \subset D$, $\exists$ $T < T_+$ s.t. $(t, x(t)) \notin K$ for $t > T$.

→ Consider autonomous system $\begin{cases} \dot{x}(t) = f(x) \\ x(t_0) = x_0 \end{cases}$.

→ **CONTRAPOSITIVE of PREVIOUS THM:**

- Suppose f is Loc. Lip. on $U \subset \mathbb{R}^n$, open. Then, if $x(t)$ stays in a COMPACT set $C \subset U$ for all $t < T_+$, then $T_+ = \infty$.

→ **EX:** If have **bound** $|x(t)| \leq e^t |x(0)|$, where $|\cdot|$ is norm of closed space, suppose $T_+ < \infty$.

But $|x(t)| \leq e^{T_+} |x(0)|$, $\forall t < T_+$,

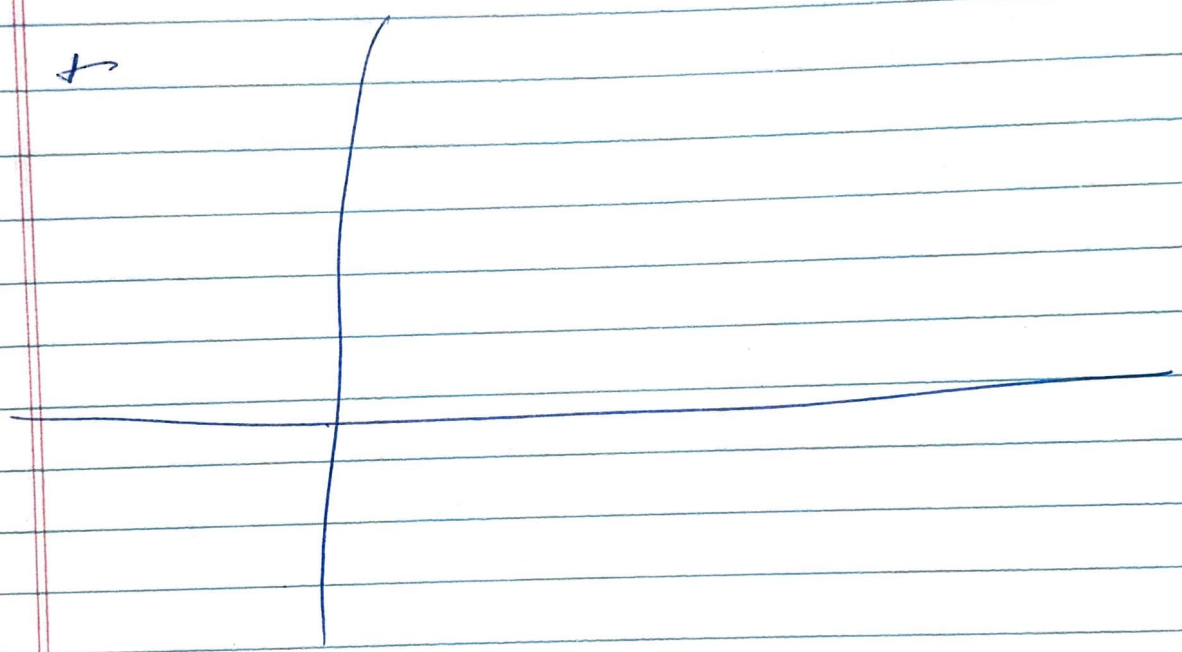
i.e., $|x(t)| \in \{ x \in \mathbb{R}^n : |x| \leq e^{T_+} |x(0)| \}$

(closed in **compact**) $\forall t < T_+$!!

$\Rightarrow$ $T_+ = \infty$.

$$|u(t)|_X \leq e^t |u(0)|_X$$

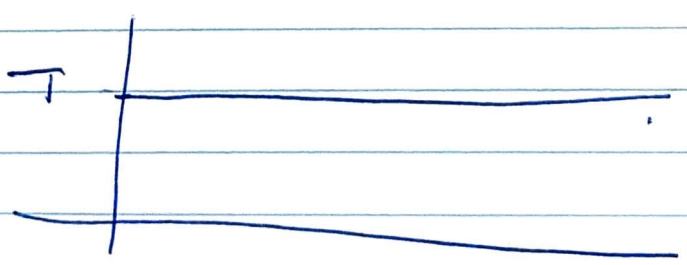
$$|u(t)|_X \leq e^{t-s} |u(s)|_X \leftarrow \text{BAD}$$



$$u_{tt} - \Delta u_t + u^3 = 0$$

$$|u|_{H^1} \sim e^t |u(0)|_{H^1} \leftarrow \text{GOOD!!}$$

$\leftarrow$ global bad



$$y(t) \leq e^{t} y(0) + \int_{T}^{t} (\quad)$$

$y(T)$

$$|t - T| \leq \frac{1}{2} |e^{T} y(0)|$$

$$y(t) \leq 2|y(0)| \qquad 0 < t \leq T$$

T

$$y(t) \leq 2|y(T)|$$

$$y(t) \leq e^{t} y(0) \qquad 0 < t < \forall t$$