• RELLICH COMPACTNESS :

$U \subset \mathbb{R}^n$ open, bounded subset, ∂U smooth boundary. Suppose $1 \leq p < n$. Then,

$$W^{1,p}(U) \subset\subset L^q(U),$$

$\forall \quad 1 \leq q < p^*$.

. REMARK: Since U is BDD, by Hölder

$$W^{1,n}(U) \subset W^{1,p}(U),$$

for $1 \leq p < n$.

Notice that $p^* \longrightarrow \infty$ as $p \longrightarrow n$. Thus, choose p suff. close to n so that $p^* > n$. Then,

$$W^{1,n}(U) \subset W^{1,p}(U) \subset\subset L^n(U).$$

or $$W^{1,n}(U) \subset W^{1,p}(U) \subset\subset L^q(U),$$

$\forall\, q \in [1, p^*]$, $p^* > n$.

- Moreover, since $p^* \to \infty$ as $p \to n$, $n > p$, then can always choose p so close to n so that ($p^* >> n$)

$$W^{1,n}(U) \subset W^{1,p}(U) \subset\subset L^q(U),$$

$$\forall \quad 1 \leq q < \infty.$$

$\Rightarrow$ (i): $\boxed{W^{1,n}(U) \subset\subset L^n(U)}$

(ii): $\boxed{W^{1,n}(U) \subset\subset L^q(U), \ \forall 1 \leq q < \infty}$

- Further, for $1 \leq p < n$, $p^* > p$

$$\Rightarrow W^{1,p}(U) \subset\subset L^p(U).$$

for $p = n$, have (i) above. For $p_1 > n$, choose $p^{(<n)}$ so close to n so that $p^* > p_1$. Then,

$$W^{1,p_1}(U) \subset W^{1,p}(U) \subset\subset L^{p_1}(U),$$

since $p_1 < p^*$. $\Rightarrow\Rightarrow$

$\Rightarrow$ (iii): $\boxed{W^{1,p}(U) \subset\subset L^p(U), \quad \forall\, 1 \leq p < \infty}$

- **WARNING**: $W^{1,p}(U) \subset [redacted]^{p^*}(U)$

BUT:

!! $\boxed{W^{1,p}(U) \subset \not\subset [redacted]^{p^*}(U)}$
